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Short communication

A frame-invariant formulation of Fung elasticity

Gerard A. Ateshian [a,b,*], Kevin D. Costa [b]

[a] Department of Mechanical Engineering, Columbia University, New York, NY, USA
[b] Department of Biomedical Engineering, Columbia University, New York, NY, USA

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ABSTRACT

Fung elasticity refers to the hyperelasticity constitutive relation proposed by Fung and co-workers for describing the pseudo-elastic behavior of biological soft tissues undergoing finite deformation. A frame-invariant formulation of Fung elasticity is provided for material symmetries ranging from orthotropy to isotropy, which uses Lamé-like material constants. In the orthotropic case, three orthonormal vectors are used to define mutually orthogonal planes of symmetry and associated texture tensors. The strain energy density is then formulated as an isotropic function of the Lagrangian strain and texture tensors. The cases of isotropy and transverse isotropy are derived from the orthotropic case. Formulations are provided for both material and spatial frames. These formulations are suitable for implementation into finite element codes. It is also shown that the strain energy function can be naturally uncoupled into a dilatational and a distortional part, to facilitate the computational implementation of incompressibility.

1. Introduction

The constitutive relation proposed by Fung et al. (1979) and Fung (1993) for the pseudo-elastic behavior of soft biological tissues has been used extensively (Sacks and Sun, 2003). The strain energy density function for this hyperelastic constitutive relation may be expressed in the form

$$W = \tfrac{1}{2}c(e^{Q} - 1) \tag{1}$$

where

$$Q = \tfrac{1}{2}a_{KLMN}E_{KL}E_{MN} \tag{2}$$

c is a material coefficient with units of stress, a_{KLMN} are dimensionless material parameters, E_{KL} are components of the Green–Lagrange strain tensor $\mathbf{E}$ (Holzapfel, 2000), and summations over $K, L, M, N = 1, 2, 3$ are implicit (Humphrey, 2002). Based on the form of these relations and the symmetry of $\mathbf{E}$, it follows that a_{KLMN} represents a set of at most 21 distinct material constants.

Implementing this finite deformation constitutive relation into a 3D finite element formulation can be achieved more conveniently when using a frame-invariant formulation. This frame-invariant form is presented here for orthotropic, transversely isotropic, and isotropic symmetry, in material and spatial frames.

2. Material and spatial frames

The second Piola–Kirchhoff stress is given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \frac{1}{2}ce^{Q}\frac{\partial Q}{\partial \mathbf{E}} \tag{3}$$

and the material elasticity tensor is[1]

$$\begin{aligned}\mathbb{C} &= \frac{\partial \mathbf{S}}{\partial \mathbf{E}} = \frac{\partial^2 W}{\partial \mathbf{E}^2}\\ &= \frac{1}{2}ce^{Q}\left(\frac{\partial Q}{\partial \mathbf{E}} \otimes \frac{\partial Q}{\partial \mathbf{E}} + \frac{\partial^2 Q}{\partial \mathbf{E}^2}\right)\\ &= 2c^{-1}e^{-Q}\mathbf{S}\otimes\mathbf{S} + \frac{1}{2}ce^{Q}\frac{\partial^2 Q}{\partial \mathbf{E}^2}\end{aligned} \tag{4}$$

The Cauchy stress is obtained from $\mathbf{S}$ using the Piola transformation (Bonet and Wood, 1997),

$$\mathbf{T} = J^{-1}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T = \frac{1}{2}J^{-1}ce^{Q}\mathbf{F}\cdot\frac{\partial Q}{\partial \mathbf{E}}\cdot\mathbf{F}^T \tag{5}$$

* Corresponding author at: Department of Mechanical Engineering, Columbia University, 500 West 120th Street, MC4703, New York, NY 10027, USA.
E-mail address: ateshian@columbia.edu (G.A. Ateshian).

[1] The tensor dyadic products $\otimes$, $\underline{\otimes}$ and $\overline{\underline{\otimes}}$ are described by Curnier et al. (1994): $(\mathbf{M}\otimes\mathbf{N})_{ijkl} = M_{ij}N_{kl}$, $(\mathbf{M}\underline{\otimes}\mathbf{N})_{ijkl} = M_{ik}N_{jl}$, $(\mathbf{M}\overline{\underline{\otimes}}\mathbf{N})_{ijkl} = \frac{1}{2}(M_{ik}N_{jl} + M_{il}N_{jk})$ for arbitrary second-order tensors $\mathbf{M}$ and $\mathbf{N}$.

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where $\mathbf{F}$ is the deformation gradient and $J = \det \mathbf{F}$. Similarly, the spatial elasticity tensor is given by

$$\begin{aligned}\mathscr{C} &= J^{-1}(\mathbf{F}\underline{\otimes}\mathbf{F}) : \mathbb{C} : (\mathbf{F}^T \underline{\otimes} \mathbf{F}^T) \\ &= 2Jc^{-1}e^{-Q}\mathbf{T}\otimes\mathbf{T} \\ &\quad + \frac{1}{2}J^{-1}ce^{Q}(\mathbf{F}\underline{\otimes}\mathbf{F}) : \frac{\partial^2 Q}{\partial \mathbf{E}^2} : (\mathbf{F}^T \underline{\otimes} \mathbf{F}^T)\end{aligned} \tag{6}$$

3. Frame-invariant forms

3.1. Orthotropic symmetry

For orthotropic and higher symmetries, we can express the strain energy density in a frame-invariant form as

$$Q = c^{-1}\sum_{a=1}^{3}\left[2\mu_a \mathbf{A}_a^0 : \mathbf{E}^2 + \sum_{b=1}^{3}\lambda_{ab}(\mathbf{A}_a^0 : \mathbf{E})(\mathbf{A}_b^0 : \mathbf{E})\right] \tag{7}$$

where

$$\mathbf{A}_a^0 = \mathbf{a}_a^0 \otimes \mathbf{a}_a^0 \tag{8}$$

In this expression, $\mathbf{a}_a^0$ are the preferred directions of material texture in the reference configuration, representing unit normal vectors to the orthotropic planes of symmetry, and satisfying $\mathbf{a}_a^0 \cdot \mathbf{a}_b^0 = \delta_{ab}$. The associated tensor $\mathbf{A}_a^0$ given in the above equation may be called texture tensors in the reference configuration. Note that $\lambda_{ba} = \lambda_{ab}$, which implies that there are six distinct coefficients λ_{ab} in this expression. Thus there are 10 material constants in this constitutive relation (c, λ_{ab}, μ_a), all having units of stress.

Example 1. Consider texture vectors aligned with the basis vectors, $\mathbf{a}_a^0 = \mathbf{e}_a$, then $\mathbf{A}_a^0 : \mathbf{E} = (\mathbf{e}_a \otimes \mathbf{e}_a) : \mathbf{E} = E_{aa}$ (no sum implied) and

$$\begin{aligned}Q = c^{-1}[&(\lambda_{11}+2\mu_1)E_{11}^2 + (\lambda_{22}+2\mu_2)E_{22}^2 + (\lambda_{33}+2\mu_3)E_{33}^2 \\ &+ 2\lambda_{23}E_{22}E_{33} + 2\lambda_{31}E_{33}E_{11} + 2\lambda_{12}E_{11}E_{22} \\ &+ 2(\mu_2+\mu_3)E_{23}E_{32} + 2(\mu_3+\mu_1)E_{13}E_{31} \\ &+ 2(\mu_1+\mu_2)E_{12}E_{21}]\end{aligned} \tag{a}$$

This expression can be used to find the equivalence between the material coefficients c, λ_{ab} and μ_a used here, and those employed in other notations. For example, Humphrey (2002) uses

$$\begin{aligned}Q = &c_1E_{RR}^2 + c_2E_{\Theta\Theta}^2 + c_3E_{ZZ}^2 \\ &+ 2c_4E_{RR}E_{\Theta\Theta} + 2c_5E_{\Theta\Theta}E_{ZZ} + 2c_6E_{ZZ}E_{RR} \\ &+ c_7(E_{R\Theta}^2 + E_{\Theta R}^2) + c_8(E_{\Theta Z}^2 + E_{Z\Theta}^2) \\ &+ c_9(E_{ZR}^2 + E_{RZ}^2)\end{aligned} \tag{b}$$

in a cylindrical coordinate system. Letting the cylindrical basis $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z\}$ be equivalent to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we find that

$$\begin{aligned}&c_1 = c^{-1}(\lambda_{11}+2\mu_1), \quad c_2 = c^{-1}(\lambda_{22}+2\mu_2), \\ &c_3 = c^{-1}(\lambda_{33}+2\mu_3), \quad c_4 = c^{-1}\lambda_{12}, \\ &c_5 = c^{-1}\lambda_{23}, \quad c_6 = c^{-1}\lambda_{13} \\ &c_7 = c^{-1}(\mu_1+\mu_2), \quad c_8 = c^{-1}(\mu_2+\mu_3) \\ &c_9 = c^{-1}(\mu_3+\mu_1)\end{aligned} \tag{c}$$

or equivalently,

$$\begin{aligned}&\mu_1 = \frac{c}{2}(c_7 - c_8 + c_9), \quad \mu_2 = \frac{c}{2}(c_7 + c_8 - c_9) \\ &\mu_3 = \frac{c}{2}(-c_7 + c_8 + c_9) \\ &\lambda_{11} = c(c_1 - c_7 + c_8 - c_9), \quad \lambda_{22} = c(c_2 - c_7 - c_8 + c_9) \\ &\lambda_{33} = c(c_3 + c_7 - c_8 - c_9) \\ &\lambda_{12} = cc_4, \quad \lambda_{23} = cc_5, \quad \lambda_{13} = cc_6\end{aligned} \tag{d}$$

Humphrey (2002) has summarized values of c and c_1–c_6 from various experimental studies on arteries and noted that only one study (Deng et al., 1994) has reported a representative value for the shear moduli (related to c_7–c_9).

From the frame-invariant representation of Eq. (7) it follows that

$$\frac{\partial Q}{\partial \mathbf{E}} = c^{-1}\sum_{a=1}^{3}\left[2\mu_a(\mathbf{A}_a^0\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{A}_a^0) + \sum_{b=1}^{3}\lambda_{ab}[(\mathbf{A}_a^0:\mathbf{E})\mathbf{A}_b^0 + (\mathbf{A}_b^0:\mathbf{E})\mathbf{A}_a^0]\right] \tag{9}$$

and

$$\frac{\partial^2 Q}{\partial \mathbf{E}^2} = c^{-1}\sum_{a=1}^{3}\left[2\mu_a(\mathbf{A}_a^0\underline{\overline{\otimes}}\mathbf{I} + \mathbf{I}\underline{\overline{\otimes}}\mathbf{A}_a^0) + \sum_{b=1}^{3}\lambda_{ab}(\mathbf{A}_a^0\otimes\mathbf{A}_b^0 + \mathbf{A}_b^0\otimes\mathbf{A}_a^0)\right] \tag{10}$$

Thus, using Eq. (9) in Eq. (3), the second Piola–Kirchhoff stress is given by

$$\mathbf{S} = e^{Q}\sum_{a=1}^{3}\left[\mu_a(\mathbf{A}_a^0\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{A}_a^0) + \frac{1}{2}\sum_{b=1}^{3}\lambda_{ab}[(\mathbf{A}_a^0:\mathbf{E})\mathbf{A}_b^0 + (\mathbf{A}_b^0:\mathbf{E})\mathbf{A}_a^0]\right] \tag{11}$$

Similarly, using Eq. (10) in Eq. (4) produces the material elasticity tensor

$$\mathbb{C} = 2c^{-1}e^{-Q}\mathbf{S}\otimes\mathbf{S} + e^{Q}\sum_{a=1}^{3}\left[\mu_a(\mathbf{A}_a^0\underline{\overline{\otimes}}\mathbf{I} + \mathbf{I}\underline{\overline{\otimes}}\mathbf{A}_a^0) + \frac{1}{2}\sum_{b=1}^{3}\lambda_{ab}(\mathbf{A}_a^0\otimes\mathbf{A}_b^0 + \mathbf{A}_b^0\otimes\mathbf{A}_a^0)\right] \tag{12}$$

To evaluate the stress and elasticity tensor in the spatial frame, recognize that upon deformation the vectors $\mathbf{a}_a^0$ transform to $\mathbf{F}\cdot\mathbf{a}_a^0 = \lambda_a\mathbf{a}_a$, where λ_a is the stretch along $\mathbf{a}_a^0$, and $\mathbf{a}_a$ is a unit vector. Thus $\lambda_a^2 = (\mathbf{F}\cdot\mathbf{a}_a^0)\cdot(\mathbf{F}\cdot\mathbf{a}_a^0) = \mathbf{A}_a^0 : \mathbf{C} \equiv K_a$ where $\mathbf{C} = \mathbf{F}^T\cdot\mathbf{F}$ is the right Cauchy–Green tensor. We can similarly define $L_a \equiv \mathbf{A}_a^0 : \mathbf{C}^2$ so that

$$\begin{aligned}\mathbf{A}_a^0 : \mathbf{E} &= \tfrac{1}{2}(K_a - 1) \\ \mathbf{A}_a^0 : \mathbf{E}^2 &= \tfrac{1}{4}(L_a - 2K_a + 1)\end{aligned} \tag{13}$$

where we used the identities $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ and $\mathbf{A}_a^0 : \mathbf{I} = 1$. Now the expression for Q in Eq. (7) can be rewritten as

$$Q = \frac{1}{4}c^{-1}\sum_{a=1}^{3}\left[2\mu_a(L_a - 2K_a + 1) + \sum_{b=1}^{3}\lambda_{ab}(K_a - 1)(K_b - 1)\right] \tag{14}$$

Substituting Eq. (13) into Eq. (11), and the resulting expression into Eq. (5) yields

$$\mathbf{T} = \frac{1}{2}J^{-1}e^{Q}\left[\sum_{a=1}^{3}\mu_aK_a[\mathbf{A}_a\cdot(\mathbf{B}-\mathbf{I}) + (\mathbf{B}-\mathbf{I})\cdot\mathbf{A}_a] + \frac{1}{2}\sum_{b=1}^{3}\lambda_{ab}[(K_a-1)K_b\mathbf{A}_b + (K_b-1)K_a\mathbf{A}_a]\right] \tag{15}$$

where $\mathbf{B} = \mathbf{F}\cdot\mathbf{F}^T$ is the left Cauchy–Green tensor and $\mathbf{A}_a = K_a^{-1}\mathbf{F}\cdot\mathbf{A}_a^0\cdot\mathbf{F}^T = \mathbf{a}_a\otimes\mathbf{a}_a$. The elasticity tensor in the spatial