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Regular and chaotic growth in a Hicksian floor/ceiling model

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ABSTRACT

In some previous papers the present authors reassembled the Hicksian trade cycle model in a new way. The floor was tied to depreciation on capital, itself the cumulative sum of past net investments, for which the principle of acceleration provided an explanation. Hence no alien elements were needed to include capital, and so close the system. The resulting model created a growth trend along with growth rate cycles, which could be periodic or quasiperiodic. In the current paper, the ceiling, using capital stock as a capacity limit for production, is added. It then turns out that pure growth no longer exists, and chaos and multistability become possible, which they were not in the previous model. A variety of bifurcation scenarios is explored, and a full understanding of the working of the four-piece, originally three-dimensional, piecewise smooth map, is attained, using a reduction to a one-dimensional return map.

1. Introduction

1.1. Background

In this paper the Hicksian floor–ceiling model of the business cycle is reconsidered. As we know, the multiplier-accelerator model invented by Samuelson (1939) and also used by Hicks (1950) could produce either pure growth or cycles, depending on the parameters. In Samuelson's linear format the model (whether the solution was cyclic or growing/declining) would either explode or else extinguish any initially present motion. The Hicksian floor and ceiling, which made the model non-linear, introduced the possibility of sustained cyclic variation.

However, there could still be *either* growth or cycles, never both. Therefore Hicks added an *exogenous* growth trend in terms of autonomous expenditures, around which the multiplier-accelerator model produced sustained cycles bounded by the floor (lower bound to disinvestment) and ceiling (upper capacity bound to production).

In Puu et al. (2005) the possibility of making growth endogenous was explored. As the floor is a limit to disinvestment, representing the decrease in the capital stock occurring when no worn out capital at all was replaced, it was suggested to link it to the actual capital stock through an assumed rate of depreciation. Capital was not a variable in Hicks's original model, but, as capital is nothing but the cumulative sum of its increments, i.e., investments, for which there was a theory in the model, no alien elements were needed for inclusion of the capital variable, thus closing the model. The advantage was that the model now created an *endogenous growth* trend along with *cycles in the growth rates*. At the same time the growth rates in the cyclic regimes, which had been enormous in the original model, were reduced to but a fraction, which adds one feature of realism to the, still admittedly, much oversimplified model.

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Growing models are not readily analyzed by standard methods, so the model in Puu et al. (2005) was converted to new "relative" variables, the growth factor and the capital to income ratio. This, however, made the original piecewise linear model non-linear, but also had the advantage of reducing its dimension from three to two.

The model with floor only, retained the unrealistic feature that with high accelerators it still created pure exponential growth. Therefore, in Puu (2007) a ceiling along Hicks's original line of thought was added. The simplest argument behind the principle of acceleration is a fixed coefficient limitational production technology that makes producers use capital and other inputs in strict proportion to the output. The same idea could, however, also be used for an upper limit (ceiling) to possible output (=real income), again making use of the capital variable. In Puu (2007) it was proved that no more pure growth points did exist as possible trajectories of the model. However, although producing some exemplifying trajectories and bifurcation diagrams, which hinted at the possibility of complex results (such as chaos), the model remained mathematically largely unexplored. To carry out such a mathematical analysis is the objective of the present paper.

1.1.1. *The Samuelson model*

"Multiplier theory" assumes consumption to be a given fraction, c, of income, i.e., $C_t = cY_{t-1}$, where a one period lag is allowed for consumption out of incomes earned the preceding period.[1]

The "principle of acceleration" determines investment. The idea is that capital is needed in a fixed proportion, a, to the (real) income to be produced, i.e., $K_t = aY_{t-1}$, where again a one period lag is allowed for the building up of capital. As investments are the change of capital stock, i.e., $I_t = K_t - K_{t-1}$, we have $I_t = a(Y_{t-1} - Y_{t-2})$.[2]

Plugging $I_t = a(Y_{t-1} - Y_{t-2})$ and $C_t = cY_{t-1}$ into

$$Y_t = C_t + I_t, \tag{1}$$

the reduced form equation:

$$Y_t = (c + a)Y_{t-1} - aY_{t-2} \tag{2}$$

is obtained, which is Hicks's version of the original Samuelson model.[3]

1.1.2. *Hicksian floor and ceiling*

The floor: According to $I_t = a(Y_{t-1} - Y_{t-2})$ investments become negative when income is decreasing. However, disinvestment cannot exceed the natural depreciation on capital in the absence of any reinvestment to replace worn out capital. For this reason Hicks introduced a lower bound for disinvestment $-I_t^f$. Accordingly, the principle of acceleration took the form

$$I_t = \max(a(Y_{t-1} - Y_{t-2}), -I_t^f) \tag{3}$$

The Ceiling: The ceiling is motivated as follows. Given income is growing very fast, some resources may become limiting, so there is also a ceiling for income Y_t^c, and the income formation Eq. (1) is replaced by:

$$Y_t = \min(C_t + I_t, Y_t^c) \tag{4}$$

Using the consumption function $C_t = cY_{t-1}$ in (4) and substituting from (3), we obtain the new reduced form:

$$Y_t = \min(cY_{t-1} + \max(a(Y_{t-1} - Y_{t-2}), -I_t^f), Y_t^c) \tag{5}$$

which replaces (2).

1.1.3. *Capital formation*

Before leaving the original model let us check that it contained a capital formation theory, though it was never stated explicitly, as it did not feed back into the original model. From the definition of investments $I_t = K_t - K_{t-1}$, we have

$$K_t = K_{t-1} + I_t$$

so, using (3),

$$K_t = K_{t-1} + \max(a(Y_{t-1} - Y_{t-2}), -I_t^f) \tag{6}$$

which is an updating equation for capital.

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[1] The accounting identity reads $Y_t = C_t + I_t$, or $Y_t = cY_{t-1} + I_t$. In equilibrium, $Y_t = Y_{t-1}$, so $Y_t = (1/1 - c)I_t$. As $c < 1$, the factor $(1/1 - c) > 1$ multiplies up investment expenditures, whence the term multiplier.

[2] Assume an aggregate production function $\bar{Y}_t = \min(\bar{K}_{t-1}/a, \bar{L}_{t-1}/b)$, where $\bar{K}_{t-1}$ and $\bar{L}_{t-1}$ denote *planned* capital and labour, needed for producing output $\bar{Y}_t$, *expected* to be demanded at the end of period t. Further, assume a naive forecasting rule $\bar{Y}_t = Y_{t-1}$, that the formation of capital takes exactly one period, i.e. $K_t = \bar{K}_{t-1}$, and that labour supply is affluent and it hence is the availability of capital that is binding. We have $K_t = aY_{t-1}$.
According to definition, investment becomes $I_t = K_t - K_{t-1} = a(Y_{t-1} - Y_{t-2})$, which justifies the principle of acceleration.

[3] To be quite true to history, Samuelson (1939) applied the accelerator to consumption only, i.e., assumed $I_t = a(C_{t-1} - C_{t-2})$. It was Hicks (1950) that replaced consumption by income.