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Exact solutions of the vorticity equation on the sphere as a manifold

ISMAEL PÉREZ-GARCÍA
Centro de Ciencias de la Atmósfera, Universidad Nacional Autónoma de México, Circuito de la Investigación Científica s/n, Ciudad Universitaria, 04510 México, D.F.
E-mail: ismael@unam.mx

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RESUMEN

El propósito de este trabajo es representar las soluciones exactas de la ecuación de vorticidad barotrópica sobre la esfera unitaria S^2 en rotación como una variedad, que son flujos zonales, ondas Rossby-Haurwitz y soluciones generalizadas llamadas modones. Se relacionan los métodos modernos de la teoría de funciones con la esfera definida como una variedad compacta y diferenciable. Cuando ésta se ha comprendido de forma correcta, se esclarece la noción abstracta de mapa local, cambio de mapa y atlas. Uno de los objetivos de este trabajo es entender mejor la solución de la ecuación de vorticidad barotrópica sobre la variedad S^2 y su utilidad para identificar las propiedades de las soluciones en la variedad Riemanniana (S^2, g). Por lo tanto, estará disponible un tipo más general de espacio que también puede contener información geométrica y analítica sustancial sobre las soluciones a la ecuación de vorticidad barotrópica.

ABSTRACT

The purpose of this paper is to represent the exact solutions of the barotropic vorticity equations (BVE) on the rotating unit sphere S^2 as a manifold, which are zonal flows, Rossby-Haurwitz waves and generalized solutions named modons. Modern methods of the function theory are connected to the sphere defined as a compact differentiable manifold. When the differentiable manifold S^2 is well understood, the abstract notion of local chart, change of chart, and atlases becomes evident. One of the aims of this paper is to better understand the solution of the barotropic vorticity equation on the manifold S^2 and its usefulness to identify the properties of the solutions on the Riemannian manifold (S^2, g). Therefore, a more general type of space will be available, which can also contain substantial geometric and analytic information about solutions for the barotropic vorticity equation.

Keywords: Rossby-Haurwitz waves, modons, hydrodynamics equation on manifolds, unit sphere, mathematical analysis of barotropic model.

1. Introduction

Let $S^2 = \{x \in R^3 : |\, x \,| = 1\}$ denote the unit sphere in R^3. The large-scale dynamics of the atmosphere on the rotating sphere S^2 can approximately be governed by the non-linear barotropic vorticity equation (BVE), which can be written in the non-dimensional form as:

$$\frac{\partial \Delta \Psi}{\partial t} + J\left(\Psi,\ \Delta \Psi + 2\mu\right) = 0 \qquad (1)$$

where $\Psi(\lambda, \mu)$ denotes the stream function, $\mu = \sin \phi = \cos\theta$, $-\pi \le \lambda \le \pi$, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$, $0 < \theta < \pi$, λ the longitude, ϕ the latitude, and θ the colatitude. Δ is the Laplace-Beltrami operator on a sphere and $J(\Psi, h)$ is the Jacobian.

The following is a solution for Eq. (1) on the sphere proposed by Thompson (1982):

$$\Psi(\lambda, \mu, t) = Y_\nu\,(\lambda', \mu') - \omega\mu + D \qquad (2)$$

where (λ', μ') are the spherical coordinates relative to a rotated pole N' with coordinates (λ_0, μ_0) with respect to the original system, and Y_ν is an eigenfunction of the operator Laplace-Beltrami with eigenvalue χ_ν. Verkley(1984) generalized Thompson's solution and demonstrated that Y_ν could be a set of eigenfunctions that contains more than only spherical harmonics. Then Eq. (2) describes a configuration in which the structure Y_ν moves through the zonal flow $-\omega\mu$ with constant velocity c_ν and without changing size and shape. The pole of the primed system N' that moves along a latitude at a constant angular velocity c_ν is given by

$$c_\nu = \omega - \frac{2(\omega + 1)}{\chi_\nu} \tag{3}$$

where χ_ν is an eigenvalue for the spectral problem $\Delta Y_\nu = -\chi_\nu Y_\nu$. In particular, for spherical harmonics $Y(\lambda', \mu')$ of degree n corresponding to the eigenvalue $\chi_\nu = \chi_n = n(n+1)$, Eq.(2) is a Rossby-Haurwitz (RH) wave. RH waves have proven to be very useful to describe the large-scale wave structure of atmospheric circulation in middle latitudes (Rossby, 1939; Haurwitz, 1940). The solution modon is constructed to divide the sphere S^2 into two regions (Tribbia, 1984; Verkley, 1984, 1987, 1990; Neven, 1992): an inner region S_i centered around the pole N', and an outer region S_o separated from the inner region by a boundary circle in which Ψ, q and its normal derivative Ψ' are continuous. Modons are considered appropriate to describe some types of atmospheric blocking events (Verkley, 1990).

Hydrodynamic equations on manifolds were studied by Ebin and Marsden (1970), Szeptycki (1973a, b), Avez and Bamberger (1978), Ghidaglia *et al.* (1988), Temam (1987) and Ilyin (1993). The existence, unicity and regularity of the solution for the evolution equation (Eq. (1)) on S^2 were proven by Szeptycki (1973a, b), Avez and Bamberger (1978), Ilyin (1993) and Skiba (2012). Ebin and Marsden (1970) dealt with the motion of an incompressible fluid on manifolds under a differential geometric point of view. Problems from the transition map between the charts are transferred to those of finding geodesics on the group of all volume-preserving diffeomorphisms, to which the methods of global analysis and infinite-dimensional geometry can be applied.

In this paper we study the manifolds S^2 in terms of the stream function Ψ for an RH wave which is sufficiently smooth and for Wu-Verkley waves and modons which are weakly differentiable of higher orders. Section 2 deals with the compact differentiable manifold S^2 and the way in which functions are constructed on this manifold. Section 3 shows the types of solutions that will be considered. Another aim of this paper is to deepen the understanding of the BVE solution on the manifold S^2 and its usage for deriving the properties of solutions to the manifold (S^2, g). The paper concludes with a summary in section 4.

2. Structure of functions on the manifold S^2

In this section we review some basic facts concerning to the manifold S^2. We should recall that the unit sphere S^2 is a compact and connected differentiable manifold. Indeed, because S^2 is compact it is not possible to cover it with only one chart. A chart of S^2 is then a pair (Ω, φ) where Ω is an open subset of S^2, and φ is a homeomorphism of Ω onto some open subset of R^2. Let us consider the two charts $\{(\Omega_\iota, \varphi_\iota), (\Omega_\kappa, \varphi_\kappa)\}$ of class C^p for S^2 where every chart corresponds to a geographical coordinate group. It is possible to define a coordinate chart that covers most of S^2 by using the standard spherical coordinate map. Let φ_ι denote the coordinate function, which maps from (x_1, x_2, x_3) to angles (λ, θ) or to (λ, μ). The domain of φ_ι^{-1} is the open set defined by $\lambda \in (-\pi, \pi)$ and $\theta \in (0, \pi)$ (this excludes the poles). The inverse map φ_ι^{-1} yields the parameterization $x_1 = \cos\lambda \sin\theta$, $x_2 = \sin\lambda \sin\theta$, $x_3 = \cos\theta$ and its variation φ_κ^{-1} yields the parameterization $\varphi_\kappa^{-1}(\lambda', \theta')=(\cos\lambda' \sin\theta', \cos\theta', \sin\lambda' \sin\theta')$. The domain of φ_κ^{-1} in the open set defined by $\lambda' \in (-\pi, \pi)$ and $\theta' \in (0, \pi)$. The charts $(\Omega_\iota, \varphi_\iota)$ and $(\Omega_\kappa, \varphi_\kappa)$ correspond to poles N and N′ on the sphere S^2. N′ might be taken as the point $(\lambda_0 = -\frac{\pi}{2}, \phi_0 = 0)$ in the old system and as the angle λ' in this new north pole, so that the new international date line is the half circle $\Gamma_\kappa = \{p \in S^2: -\frac{\pi}{2} < \lambda(p) < \frac{\pi}{2}, \theta = \frac{\pi}{2},\}$ of the old equator in the x_1x_2-plane, on the front where $x_1 \geq 0$ (Richtmyer, 1981; Skiba, 1989; Pérez-García, 2001). The international date line, for the chart $(\Omega_\iota, \varphi_\iota)$ is the half circle $\Gamma_\iota = \{p \in S^2: -\frac{\pi}{2} < \phi(p) < \frac{\pi}{2}, \lambda = \pm\pi\}$ in the x_1x_3–plane. The chart $(\Omega_\iota, \varphi_\iota)$ covers the sphere except for the set Γ_ι, and the chart $(\Omega_\kappa, \varphi_\kappa)$ similarly covers the sphere with the exception of a set Γ_κ. Hence the two charts $\{(\Omega_\iota, \varphi_\iota), (\Omega_\kappa, \varphi_\kappa)\}$ together cover S^2 and they constitute an atlas.