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Vaccination threshold size and backward bifurcation of SIR model with state-dependent pulse control

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ABSTRACT

Depending on the potential susceptible human size, we consider the state-dependent integrated infectious disease control strategies including vaccination, isolation and treatment. Correspondingly, we propose a state-dependent pulse SIR model, in which whether the control measures implemented or not depends on the threshold size of susceptible population. By defining the Poincaré map, we first investigate the existence and global stability of the semi-trivial (or disease free) periodic solution, and the threshold condition is proposed. Further, by employing bifurcation theories of the one-parameter family of maps related to the Poincaré map, we then focus on the bifurcation with respect to the key parameters. The main results reveal that backward bifurcation via transcritical bifurcation or pitchfork bifurcation can occur for all the interesting parameters including isolation rate, vaccination rate, threshold susceptible population size and birth rate. The complex relationships between the basic reproduction number of classical SIR model and the threshold condition of the model with state-dependent pulse control depict that the control strategies related to the four parameters should be carefully designed, otherwise the paradoxical effects could occur and the gains cannot make up for losses. For example, too small vaccination rate will result in an increasing of threshold condition and the number of infected population. Therefore, our results suggest that when the state-dependent feedback control strategy is implemented for infectious disease control, the effective and optimal control program should take the population dynamics, the threshold susceptible population size, vaccination and isolation or treatment rate into consideration.

1. Introduction

Infectious disease remains a key healthy issue for humans, and it has a significant impact on social stability and economic development. Mathematical analysis and modelling, as an important part of infectious disease epidemiology, has been central to infectious disease epidemiology since its inception as a discipline more than a century ago (Dietz and Heesterbeek, 2002; Grassly and Fraser, 2008). In the literatures of modelling infectious disease, the classical SIR model is usually used to describe the transmission dynamics of infectious diseases among humans, which gives:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta SI - \delta S, \\ \frac{dI(t)}{dt} = \beta SI - \gamma I - \delta I, \\ \frac{dR(t)}{dt} = \gamma I - \delta R, \end{cases} \tag{1}$$

where $S(t)$, $I(t)$ and $R(t)$ represent the population densities of susceptible, infected and recovered humans at time t, respectively. Here, Λ is the constant recruitment rate, δ denotes the death rate, β represents the transmission rate, and γ is the recovery rate.

Due to the recovered humans $R(t)$ cannot be infected again, the dynamics of system (1) are determined by the following equations:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta SI - \delta S, \\ \frac{dI(t)}{dt} = \beta SI - \gamma I - \delta I. \end{cases} \tag{2}$$

Vaccination is one of the main methods to control the spread of infectious diseases, which has proved a powerful defence against a range of infectious diseases of humans and animals (Keeling et al., 2003). However, it becomes very complicated to find an optimal vaccination strategy if we compare the costs, outcomes, and cost-effectiveness of a vaccination program with no intervention. Many researchers have tried to investigate the transmission dynamics of infectious diseases under the control program of vaccination through mathematical modelling (Grassly and Fraser, 2008; Smith et al., 2006). Some of them assumed that vaccination

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is done continuously with continuous models being considered (Ferguson et al., 2005; Kribs-Zaleta and Velasco-Hernández, 2000), many others assumed that vaccination is carried out at the fix moments with impulsive models being proposed (Agur et al., 1993; D'Onofrio, 2002; Mailleret and Lemesle, 2009; Shulgin et al., 1998; Smith and Schwartz, 2008; Yang and Xiao, 2012; Yang et al., 2013). However, either the continuous vaccination or the pulse vaccination with fixed time period ignored the prevalence of the infectious disease and the potential size of susceptible humans. Therefore, taking costs, outcomes, and cost-effectiveness into consideration, it is more reasonable to assume that whether vaccinate the susceptible humans or not, depends on the size of the infected humans or susceptible humans. Correspondingly, the state-dependent pulse model should be proposed (Bainov and Simeonov, 1993; Simeonov and Bainov, 1989). Actually, the state-dependent impulsive model has been applied to many areas, including integrated pest management (Tang and Cheke, 2005; 2008; Tang and Pang, 2017; Tang et al., 2015b; 2015a), comprehensive tumor treatment (Panetta, 1996; 1998), and neuron systems (Touboul and Brette, 2009).

In this study, we assume that, if the number of susceptible population is below a critical size S_v, we do not carry out any control strategy. However, once the susceptible population reaches the critical size S_v, the integrated interventions including vaccination and isolation (or treatment) are carried out immediately. Thus, based on model (2), we propose the following state-dependent feedback control SIR model:

$$\begin{cases} \left.\begin{aligned} \frac{dS(t)}{dt} &= \Lambda - \beta SI - \delta S \doteq F_1(S,I), \\ \frac{dI(t)}{dt} &= \beta SI - \gamma I - \delta I \doteq F_2(S,I), \end{aligned}\right\} & S(t) < S_v, \\ \left.\begin{aligned} S(t^+) &= (1-p)S(t), \\ I(t^+) &= (1-q)I(t), \end{aligned}\right\} & S(t) = S_v. \end{cases} \tag{3}$$

Here, $p \in [0, 1]$ and $q \in [0, 1]$ denote the vaccination rate of susceptible humans and the isolation (or treatment) rate ratio of the infected humans, respectively.

The main purpose of this study is to investigate the dynamics of the proposed model, and examine the efficacy of this integrated regime for controlling the spread of the infectious disease. The rest part of the paper is organised as follows. In Section 2, we first give the basic definitions of the state-dependent impulsive model and some Lemmas on the stability of the semi-trivial (or disease free) periodic solution (STPS). In Section 3, we define the Poincaré map and analyze its main properties. In Section 4, we investigate the existence and global stability of the STPS (i.e., disease free periodic solution), and provide the threshold condition. In Section 5, the transcritical and pitchfork bifurcation bifurcations have been investigated with respect to four interesting parameters, and then the occurrence of backward bifurcations (Gumel, 2012; Hadeler and Van, 1997; Wang, 2006; Zhang and Liu, 2008) are discussed, which are crucial for infectious disease control. Finally, some important results related to the disease control and design of the optimal control measures are addressed in the last section.

2. Preliminaries

We briefly summary the necessary results used throughout this paper in this section. Consider the following generalized planar impulsive semi-dynamic system

$$\begin{cases} \frac{dx_1}{dt} = P(x_1,x_2), \quad \frac{dx_2}{dt} = Q(x_1,x_2), & if \quad \phi(x_1,x_2) \neq 0, \\ \Delta x_1 = \bar{\alpha}(x_1,x_2), \quad \Delta x_2 = \bar{\beta}(x_1,x_2), & if \quad \phi(x_1,x_2) = 0. \end{cases} \tag{4}$$

Here, $(x_1,x_2) \in R_+^2 = \{(x,y) | x \geq 0, y \geq 0\}$, $\triangle x_1 = x_1^+ - x_1$ and $\triangle x_2 = x_2^+ - x_2$. P, Q, $\bar{\alpha}$, $\bar{\beta}$ are continuous functions from R_+^2 into R. The impulsive function $H : R_+^2 \to R_+^2$ is defined as

$$\begin{aligned} H(x_1,x_2) &= (H_1(x_1,x_2), H_2(x_1,x_2) \\ &= (x_1 + \bar{\alpha}(x_1,x_2), x_2 + \bar{\beta}(x_2,y_2)) \end{aligned}$$

and $Z^+ = (x_1^+, x_2^+)$ is called an impulsive point of $Z = (x_1,x_2)$. Based on the notations and definitions presented in literatures (Bonotto and Federson, 2008; Kaul, 1990; 1994; Tang et al., 2015a), we can define the planar impulsive semi-dynamic system and an order k periodic solution of model (4). In particular, the following analogue of Poincaré criterion (Simeonov and Bainov, 1989) can be used to analyze the local stability of an order-k periodic solution.

Lemma 2.1. *The T-periodic solution $(x_1,x_2) = (\xi(t), \eta(t))$ of the system (4) is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies $|\mu_2| < 1$, where*

$$\mu_2 = \prod_{k=1}^{q} \triangle_k \exp\left[\int_0^T \left(\frac{\partial P}{\partial x_1}(\xi(t),\eta(t)) + \frac{\partial Q}{\partial x_2}(\xi(t),\eta(t))\right)dt\right]$$

with

$$\triangle_k = \frac{P_+\left(\frac{\partial \bar{\beta}}{\partial x_2}\frac{\partial \phi}{\partial x_1} - \frac{\partial \bar{\beta}}{\partial x_1}\frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_1}\right) + Q_+\left(\frac{\partial \bar{\alpha}}{\partial x_1}\frac{\partial \phi}{\partial x_2} - \frac{\partial \bar{\alpha}}{\partial x_2}\frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\right)}{P\frac{\partial \phi}{\partial x_1} + Q\frac{\partial \phi}{\partial x_2}}$$

and P, Q, $\frac{\partial \bar{\alpha}}{\partial x_1}$, $\frac{\partial \bar{\alpha}}{\partial x_2}$, $\frac{\partial \bar{\beta}}{\partial x_1}$, $\frac{\partial \bar{\beta}}{\partial x_2}$, $\frac{\partial \phi}{\partial x_1}$ and $\frac{\partial \phi}{\partial x_2}$ are calculated at the point $(\xi(\tau_k), \eta(\tau_k))$, $P_+ = P(\xi(\tau_k^+), \eta(\tau_k^+))$ and $Q_+ = Q(\xi(\tau_k^+), \eta(\tau_k^+))$. Here $\phi(x_1, x_2)$ is a sufficiently smooth function such that grad$\phi(x_1, x_2) \neq 0$, and $\tau_k (k \in N)$ is the time of the kth jump.

In order to address the bifurcation of the Poincaré map defined by system (4), we introduce the following two lemmas (Grandmont, 2008):

Lemma 2.2. *(Transcritical bifurcation). Let $G : U \times I \longrightarrow R$ define a one-parameter family of maps, where G is C^r with $r \geq 2$, and U, I are open intervals of the real line containing 0. Assume that*

$$G(0,\alpha) = 0 \text{ for all } \alpha, \frac{\partial G}{\partial x}(0,0) = 1,$$

$$\frac{\partial^2 G}{\partial x \partial \alpha}(0,0) > 0, \quad \frac{\partial^2 G}{\partial x^2}(0,0) > 0.$$

Then there are $\alpha_1 < 0 < \alpha_2$ and $\varepsilon > 0$ such that

(i) *If $\alpha_1 < \alpha < 0$, then G_α has two fixed points, 0 and $x_{1\alpha} > 0$ in $(-\varepsilon, \varepsilon)$. The origin is asymptotically stable, the other fixed point is unstable.*
(ii) *If $0 < \alpha < \alpha_2$, then G_α has two fixed points, 0 and $x_{1\alpha} < 0$ in $(-\varepsilon, \varepsilon)$. The origin is unstable, the other fixed point is asymptotically stable.*

Note that the case $\frac{\partial^2 G}{\partial x \partial \alpha}(0,0) < 0$ can be handled by making the change of parameter $\alpha \to -\alpha$, and several different cases have been shown in Fig. 1.

Lemma 2.3. *(Supercritical pitchfork bifurcation). Let G: $U \times I \to R$ be as in Lemma 2.2, except that G is C^r with $r \geq 3$, $\frac{\partial^2 G}{\partial x^2}(0,0) = 0$ and $\frac{\partial^3 G}{\partial x^3}(0,0) < 0$. Then there are $\alpha_1 < 0 < \alpha_2$ and $\varepsilon > 0$ such that*

(i) *If $\alpha_1 < \alpha \leq 0$, then G_α has a unique fixed point, $x = 0$, in $(-\varepsilon, \varepsilon)$. It is asymptotically stable.*
(ii) *If $0 < \alpha < \alpha_2$, then G has three fixed points in $(-\varepsilon, \varepsilon)$. The origin is an unstable fixed point, the two others, $x_{1\alpha} < 0 < x_{2\alpha}$, are asymptotically stable.*

Similarly, the case $\frac{\partial^2 G}{\partial x \partial \alpha}(0,0) < 0$ can be handled by making the change of parameter $\alpha \to -\alpha$ and several different cases including subcritical pitchfork bifurcation have been shown in Fig. 2.