Advances in Water Resources 119 (2018) 70–83

Contents lists available at ScienceDirect

Advances in Water Resources

journal homepage: www.elsevier.com/locate/advwatres

hp discontinuous Galerkin methods for parametric, wind-driven water wave models

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ARTICLE INFO

Keywords:
Wind-driven water waves
Discontinuous Galerkin method
Polynomial enrichment
Lake Erie

ABSTRACT

We recast the parametric, wind-driven water wave modeling paradigm into a weak form that is advantageous to discontinuous Galerkin (DG) methods and demonstrate some advantages of polynomial refinement versus standard mesh refinement. Hindcast studies performed over Lake Erie indicate that this simplified parametric approach, when paired with advanced numerics, produces similar error measures to the well-established spectral wave model known as SWAN while executing *significantly* faster in terms of CPU time.

1. Introduction

Spectral wave models, which seek to describe the high-frequency band of surface waves generated by the wind and restored to equilibrium pre-dominately by gravity (see Fig. 1), have a long history of development dating back to the 1950s; see, for example, the discussions provided by the SWAMP and WAMDI groups (The SWAMP Group 1985; WAMDI Group 1988). The essential feature that sets spectral wave models apart from other wave modeling approaches is the attempt to provide a statistical description of the wave field by modeling the evolution of the wave (variance density) spectrum rather than attempting to resolve each individual wave train of the sea-surface (known as the phase– resolving approach (Wei et al., 1995)). The goal of a spectral wave model is computational tractability; by evolving a variance density spectrum, various wave properties, such as significant wave height and period, can be obtained over large domains in a timely fashion. Some of the early spectral wave models further took advantage of the fact that (a portion of) the spectrum was observed to have a universal shape (when normalized with respect to peak frequency) that could be represented by a few parameters. This observation led to the development of a class of spectral wave models referred to as parametric (wind-sea) wave models (see, for example, Hasselmann et al., 1975) — a class of models distinct from the “traditional” discrete spectral models (Holthuijsen, 2007) that are widely used today.

The primary advantage of parametric wave models is the massive reduction in computational effort that they can afford compared to discrete spectral models. The latter directly discretize the so-called spectral energy (or *action* in the presence of ambient currents) balance equation using a sizable number of frequency (typically ≈ 30) and direction (≈ 36) “bins” at each computational grid point in geographic space (SWAN). In contrast to this, parametric wave models solve a coupled set of transport equations at each computational grid point for a small number of parameters (typically ≤ 6) that describe the wave spectrum. (The parametric modeling approach is described in more detail in Section 2.1.) Therefore, the number of discrete equations used by a parametric model will typically be two orders of magnitude smaller than the number used by a (numerically) similar discrete spectral model on the same computational grid.

This reduction in computational effort, however, comes at the cost of a reduced description of the wave field. Specifically, the wave spectrum will, in general, consist of both *wind-sea* (waves under the influence of the wind that generated them) and *swell* (waves that have escaped the influence of the generating wind) components (see Fig. 1), with only the former being able to be described parametrically. To overcome this deficiency, parametric models were often combined with discrete spectral modeling approaches for the swell components only, giving rise to so-called coupled hybrid models (Günther et al., 1979), which exhibit computational costs somewhere between the pure parametric and full discrete spectral modeling approaches.

Despite the deficiencies of the parametric modeling approach, in this paper, we show that a return to this earlier, efficient parametric approach, paired with more advanced numerics, proves to be sufficiently accurate for the wind-sea cases considered.

More specifically, we recast the parametric wind-sea modeling paradigm into a variational weak form that is advantageous to discon-

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https://doi.org/10.1016/j.advwatres.2018.04.008
Received 22 June 2017; Received in revised form 10 April 2018; Accepted 23 April 2018
Available online 2 July 2018

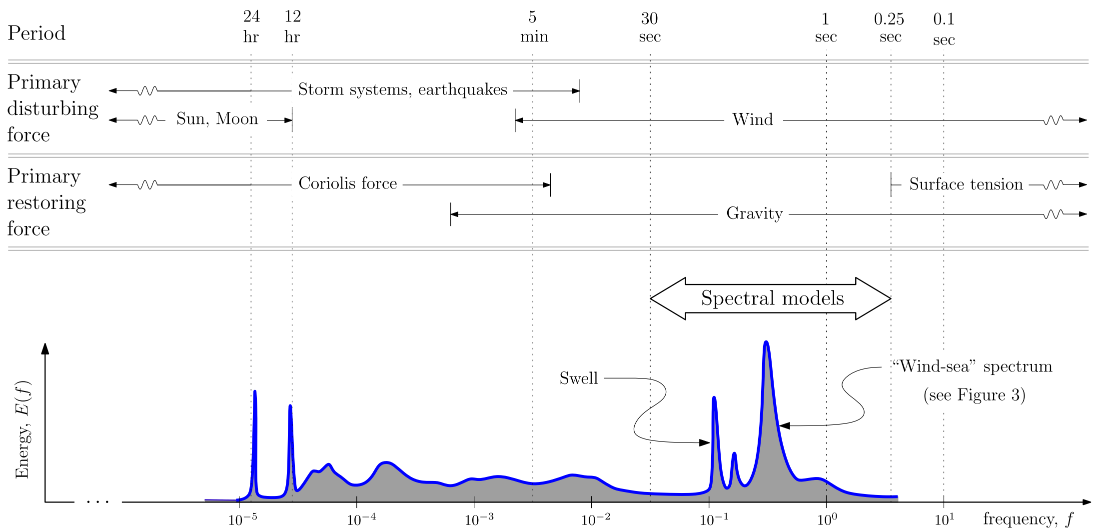

Fig. 1. Wave energy spectrum of the ocean. Note the large amount of wave energy contained within the high-frequency bands of the wind-sea and swell phenomena.

tinuous Galerkin (DG) methods. These methods are locally conservative, can handle arbitrary meshes, and can dynamically adapt in time via mesh (h) refinement and polynomial (p) refinement (Cockburn and Shu, 2001), a feature shown in Conroy et al. (2018) to be instrumental in capturing rapidly changing frequencies associated with large gusts of wind.

We further demonstrate some advantages of the DG method through a series of hindcasts over Lake Erie, where we compare model output to observational buoy data as well as hindcast data from the well-established third generation spectral wave model known as SWAN. Results indicate that this simplified parametric approach, when paired with advanced numerics, produces similar error measures to SWAN while significantly reducing the computational cost.

The remainder of this paper is organized as follows. In Section 2 we discuss general mathematical approaches to model surface gravity waves, and via a number of logical deductions, arrive at a simple yet accurate (in the proper scenario) two-parameter wave model. We recast this two-parameter wave model into a variational weak form that is suitable for the discontinuous Galerkin (DG) finite element method in Section 3 and assess the applicability of the model in Section 4 via hindcast simulations over Lake Erie. We compare numerical results to observational buoy data as well as to hindcast data from the well-established wave model known as SWAN. It is also in Section 4 where we demonstrate some advantages of the DG method in terms of using p refinement versus standard h refinement. Finally, in Section 5, we discuss some conclusions and future work.

2. Mathematical approaches

A spectral wave model aims to describe the wave environment in a statistical sense, i.e., in the context of the wave dynamics that are most likely to occur (under a given physical condition) at a specific coordinate in geographic space. Its foundation is built on the idea that an irregular sea surface can be described as a superposition of a large number of *harmonic waves*, each traveling with a distinct frequency and amplitude.

At each geographic coordinate, the spectral approach describes the sea surface (η) as a *stochastic process*, and the expected value of the variance of the sea surface, i.e., $F\langle \frac{1}{2}\eta^2\rangle$, is distributed over frequency (f) and direction (θ) components to obtain a *variance density spectrum*,

$$F(f,\theta) = \lim_{\Delta f\to 0,\ \Delta\theta\to 0} \frac{1}{\Delta f}\frac{1}{\Delta\theta} F\left\langle \frac{1}{2}\eta^2\right\rangle, \tag{1}$$

which gives a complete statistical description of the sea surface elevation if it can be seen as a *stationary*, *Gaussian* process. Moments of the spectrum, m_n, quantify wave characteristics at a given spatial coordinate,

$$m_n = \int_0^\infty \int_0^{2\pi} f^n F(f,\theta)\, d\theta\, df, \quad \text{for} \quad n = \ldots, -2, -1, 0, 1, 2, \ldots \tag{2}$$

For example, the zeroth-order moment corresponds to the variance of the surface elevation, σ^2, and is defined as

$$\sigma^2 = m_0 = \int_0^\infty F(f)\, df. \tag{3}$$

The square root of m_0 multiplied by 4 is (approximately) the so-called *significant wave height*, or design wave height for *deep water waves*, $H_{m_0} \approx 4\sqrt{m_0}$. Strictly speaking, significant wave height is defined as the average wave height of the third highest waves of the wind-sea, and is typically the most relevant wave characteristic for design purposes. Determining the design wave height of a given wave environment then merely consists of integrating the wave spectrum, and thus, quantification of the short-wave dynamics re- quires determining the shape and scale of the variance density spectrum at each coordinate in geographic space. This can be accomplished by solving a spectral energy balance equation, namely,

$$\frac{\partial F}{\partial t} + \nabla\cdot\left(c_g F\right) + \tilde{\nabla}\cdot\left(c_s F\right) = S, \tag{4}$$

where $c_{g,i} = \frac{d\omega}{dk_i}$ is the wave group velocity, $\omega = 2\pi f$ is the angular wave frequency, k_i is the wave number with $i = 1, 2, 3$, $c_{s,i} = \frac{d\omega}{dx_i}$ is the spectral wave velocity, $\tilde{\nabla} = \partial/\partial f\ \hat{e}_1 + \partial/\partial\theta\ \hat{e}_2$ is the gradient in spectral space where $\hat{e}_1$ and $\hat{e}_2$ are spectral unit vectors, and S is a source term that in general, is a function of atmospheric input, nonlinear wave-wave interactions, and dissipation. Eq. (4) can be integrated (given appropriate initial conditions and boundary conditions) to obtain $F(f, \theta; x, t)$ which gives relevant statistical information concerning the high-frequency gravity waves.

The result is a model that must be discretized in *five* dimensions—two-dimensions in geograhpic space (x, y), two-dimensions in spectral space (f, θ), and the temporal dimension. This makes the numerical discretization of (4) rather complex, and is exacerbated by calculations of the group and spectral velocities, see Dietrich et al. (2013) and Meixner et al. (2013). It is precisely this complexity that makes the numerical discretization of (4) with a DG method almost untenable in terms of full-scale "operational" applications.