Advances in Water Resources 116 (2018) 145–152

Contents lists available at ScienceDirect

Advances in Water Resources

journal homepage: www.elsevier.com/locate/advwatres

A series solution for horizontal infiltration in an initially dry aquifer

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ARTICLE INFO

Keywords:
Porous medium equation
Series solution
Groundwater
Wedged Hele–Shaw cell
Shampine's method

ABSTRACT

The porous medium equation (PME) is a generalization of the traditional Boussinesq equation for hydraulic conductivity as a power law function of height. We analyze the horizontal recharge of an initially dry unconfined aquifer of semi-infinite extent, as would be found in an aquifer adjacent a rising river. If the water level can be modeled as a power law function of time, similarity variables can be introduced and the original problem can be reduced to a boundary value problem for a nonlinear ordinary differential equation. The position of the advancing front is not known ahead of time and must be found in the process of solution. We present an analytical solution in the form of a power series, with the coefficients of the series given by a recurrence relation. The analytical solution compares favorably with a highly accurate numerical solution, and only a small number of terms of the series are needed to achieve high accuracy in the scenarios considered here. We also conduct a series of physical experiments in an initially dry wedged Hele-Shaw cell, where flow is modeled by a special form of the PME. Our analytical solution closely matches the hydraulic head profiles in the Hele-Shaw cell experiment.

1. Introduction

Groundwater flow is considered to be unconfined when it has a free upper surface in contact with a gas phase (i.e., air and water vapor), usually at atmospheric pressure. Unconfined flows are driven primarily by gravity acting on the density difference between the two phases and are considered to be a subset of a larger class of flows known as gravity-driven flows.

If the slope of the free surface is "small," the hydraulic head can be considered constant everywhere along the vertical. This is known as the Dupuit–Forchheimer (DF) assumption. In the case of primarily horizontal flow in a porous medium, the DF assumption can be used to write an expression for discharge through a control volume as a function of hydraulic head only. Depending on the properties of the porous medium, substitution of this discharge function into the continuity equation results in the Porous Medium Equation (PME), with the Boussinesq equation being a special case of particular interest to hydrologists. These equations apply to domains in which the characteristic thickness of the saturated media is much greater than that of the overlying unsaturated zone. Under these conditions, capillarity can safely be dismissed and the water-saturated part of the system can be modeled independently of the overlying unsaturated zone. Since the aqueous phase invades and under-rides a less-dense gas phase, a counter flow of gas exists in the unsaturated zone, although its effect on the flow of water is usually negligible due to the nearly three order of magnitude difference in density between liquid water and the overlying gaseous phase.

Closed-form analytical solutions for infiltration governed by the Boussinesq equation exist for a certain class of geometries and boundary conditions (Bear, 1988; Polubarinova-Kochina, 1962; Tolikas et al., 1984). For one-dimensional horizontal flow described by the Boussinesq equation in an initially dry aquifer, Lockington et al. (2000) derived a solution in which one boundary is a power law function of time. By introducing similarity variables the problem was reduced to a free boundary value problem for a nonlinear ordinary differential equation (ODE), and an approximate solution in the form of a quadratic polynomial was constructed. Olsen and Telyakovskiy (2013) extended the approach of Lockington et al. (2000) to flows governed by the generalized Boussinesq equation for the same initial and boundary conditions.

Rupp and Selker (2005) examined Boussinesq-style drainage from a fully saturated aquifer with hydraulic conductivity taken as a power law function of height. One wall of the aquifer considered is an impermeable vertical boundary, while the other vertical boundary allows instantaneous drawdown from a horizontal non-zero initial condition. It was shown that the PME results from solving the flow equation with hydraulic conductivity given as a power law function of height. Zheng et al. (2013) derived equations for the same problem in a wedged Hele-Shaw cell. Ciriello et al. (2013) investigated special cases of permeability varying in both the vertical and horizontal directions, with boundary conditions determined by injection rates.

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https://doi.org/10.1016/j.advwatres.2018.04.005
Received 15 July 2017; Received in revised form 8 April 2018; Accepted 10 April 2018
Available online 12 April 2018
0309-1708/

Zheng et al. (2014) modeled and ran an experiment for the spreading of a mound of fluid, where permeability varied as a power law in the horizontal direction. Longo et al. (2015) also investigated the case of a spreading mound, but for a non-Newtonian fluid.

Barenblatt (1952) solved the PME in the context of gas flow through porous media. A zero initial condition was used for both power law hydraulic head and power law volumetric flux inlet boundary conditions. Planar, cylindrical, and spherical symmetries were considered. It is shown that weak solutions to this class of problems possess advancing fronts, which propagate with finite speed. The solutions are obtained by the introduction of dimensionless similarity variables, which reduce the partial differential equation (PDE) to an ODE. The first few terms of a power-series solution are explicitly given, but the approach undertaken requires significant effort to generate further terms. No term-generating algorithm is provided, and the powers in the series are non-integer. Song et al. (2007) applied the approach of Barenblatt (1952) to investigate the Boussinesq equation with a power law hydraulic head inlet boundary condition for an initially dry aquifer. In addition, Song et al. (2007) obtained a recursion relation for the coefficients of the power series terms, greatly simplifying the process of approximating the solution with arbitrary accuracy. This solution was extended to a power law volumetric flux at the boundary by Telyakovskiy et al. (2010).

We approach the problem of flow in an initially dry, unconfined aquifer in a semi-infinite space, where the hydraulic head at the upstream boundary is a power law function of time and the hydraulic conductivity diminishes with depth. Such flows may occur in shallow systems where saturated groundwater invades a partially saturated zone with soil bulk density increasing with depth. We solve a class of problems similar to Barenblatt (1952) and Song et al. (2007) and provide an easy way to generate terms for the series solution. The series solution is compared to a numerical solution developed by Shampine (1973). Results are also compared to an experiment similar to those performed in Zheng et al. (2013), Ciriello et al. (2016), and the theoretical setting presented by Rupp and Selker (2005).

2. Problem statement

We provide a brief derivation of the PME in the context of one dimensional groundwater flow in the x-direction over a horizontal impervious base. We define $h^\star$ to be the height of the phreatic surface, as measured vertically from the base of the aquifer. Under the hydrostatic assumption, velocity v is given by Darcy's law,

$$v = -\frac{\rho g k}{\mu}\frac{\partial h^\star(x,t)}{\partial x}, \tag{1}$$

where k is the intrinsic permeability and μ the viscosity. Mass conservation represents a rate of change of the amount of fluid in the elementary volume. It is compensated by the fluxes through the left and right vertical faces, giving the continuity equation:

$$\frac{\partial}{\partial t}\int_x^{x+dx}\int_0^{h^\star(x,t)} \epsilon dy dx = \int_0^{h^\star(x,t)} v dy - \int_0^{h^\star(x+dx,t)} v dy. \tag{2}$$

We assume power law expressions for porosity $\epsilon = \frac{1}{r_1}y^\phi$ and permeability $k = c_1 y^n$. Neglecting higher order terms, (2) becomes,

$$\frac{\partial}{\partial t}h^{\star\phi+1}(x,t) = \frac{r_1(\phi+1)\rho g c_1}{\mu(n+1)(n+2)}\frac{\partial^2}{\partial x^2}h^{\star n+2}(x,t). \tag{3}$$

Performing a change of variables for $h^{\star\phi+1} = h$, (3) takes the typical form,

$$\frac{\partial}{\partial t}h = a\frac{\partial^2}{\partial x^2}h^m, \tag{4}$$

where $a = \frac{r_1(\phi+1)\rho g c_1}{\mu(n+1)(n+2)}$ and $m = \frac{n+2}{\phi+1}$. The power law parameters ϕ and n are chosen such that the limiting case of the heat equation is avoided:

$$\frac{n+2}{\phi+1} > 1. \tag{5}$$

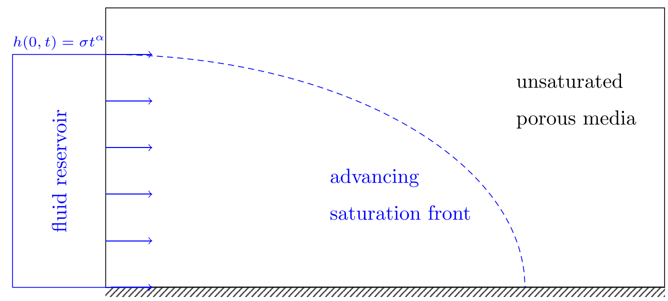

Fig. 1. A liquid, such as water, invades unsaturated porous media from a reservoir at the left boundary. The horizontal base of the aquifer is impermeable. The ambient air in the unsaturated zone is at atmospheric pressure, and has a negligible effect on the invading saturation front.

It is easily verified that for constant permeability $n = 0$ and porosity $\phi = 0$, this expression reduces to the traditional case of the Boussinesq equation. For a detailed derivation and bounds on the parameters ϕ and n, see the Supplementary Materials.

We consider (4) as the governing equation for flow from a source, such as a river or ditch, into an initially dry aquifer with an impermeable horizontal base, as shown in Fig. 1. The domain of this problem is semi-infinite and the boundary and initial conditions are specified as,

$$h(0,t) = \sigma t^\alpha, \qquad \sigma > 0, \quad -\frac{1}{m+1} \le \alpha < \infty, \tag{6}$$

$$\lim_{x\to\infty} h(x,t) = 0, \qquad t > 0, \tag{7}$$

$$h(x,0) = 0, \qquad x > 0. \tag{8}$$

The parameters σ and α are chosen to control the height of the invading fluid at the $x = 0$ inlet boundary. The water level of the fluid reservoir adjacent the aquifer is controlled by the exponent α. The value of the exponent $\alpha = -\frac{1}{m+1}$ corresponds to the free spreading of a mound of groundwater, while the case of $\alpha > 0$ corresponds to a rising water level at the boundary $x = 0$. The case of $\alpha = 1$ represents a linearly rising water level at the boundary, while the case of $\alpha = 0$ corresponds to the important practical application of a constant water level in the fluid reservoir. If $\alpha < -\frac{1}{m+1}$, then the water level at $x = 0$ drops faster than it does in the porous medium, due to gravity. This results in back seepage at $x = 0$, requiring a different analysis than the one presented here.

Depending on the properties of the fluid and porous media, the exponent m takes different values. The value $m = 5$ can be used to model flow through concretes (Lockington et al., 1999). Forest soils may have values ranging from $m = 2.2$ to $m = 8.9$ (Beven, 1982). Flow of air at atmospheric conditions through soils can be modeled with $m = 2.405$ (Vazquez, 2007). Classical groundwater flow can be modeled with the Boussinesq equation where $m = 2$, which is the case that we generalize in this paper.

The zero initial condition given by Eq. (8) represents an initially dry aquifer. Such a situation can occur when a dry river bed is suddenly flooded after a dry season, and the water level rises with $\alpha > 0$. Another important situation is the case of a river rising rapidly to a constant stage, possible during flash flooding and irrigation. For modeling purposes, this scenario can be modeled with the constant boundary condition: $h(0,t) = \sigma$, $\alpha = 0$ (Lockington, 1997). To date, multiple studies have analyzed these problems, e.g. Prasad and Salomon (2005), Lockington et al. (2000), and Srivastava et al. (2006). Similarly, the case of filtration through concretes with a zero initial saturation was considered by Lockington et al. (1999).

If the aquifer is partially filled with water initially, then the solutions to the PME propagate with infinite speed, see e.g. Polubarinova-Kochina (1962). In practical applications, we expect a finite speed of