Ocean Modelling 126 (2018) 56–62

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

Short communication

The relationship between a deformation-based eddy parameterization and the LANS-α turbulence model

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ARTICLE INFO

Keywords:
Eddy parameterization
Deformation
LANS-α
Mesoscale
Non-Newtonian fluid

ABSTRACT

A recent class of ocean eddy parameterizations proposed by Porta Mana and Zanna (2014) and Anstey and Zanna (2017) modeled the large-scale flow as a non-Newtonian fluid whose subgridscale eddy stress is a nonlinear function of the deformation. This idea, while largely new to ocean modeling, has a history in turbulence modeling dating at least back to Rivlin (1957). The new class of parameterizations results in equations that resemble the Lagrangian-averaged Navier–Stokes-α model (LANS-α, e.g., Holm et al., 1998a). In this note we employ basic tensor mathematics to highlight the similarities between these turbulence models using component-free notation. We extend the Anstey and Zanna (2017) parameterization, which was originally presented in 2D, to 3D, and derive variants of this closure that arise when the full non-Newtonian stress tensor is used. Despite the mathematical similarities between the non-Newtonian and LANS-α models which might provide insight into numerical implementation, the input and dissipation of kinetic energy between these two turbulent models differ.

1. Introduction

The problem of parameterizing ocean mesoscale eddies has received considerable attention in recent years. Many sophisticated closures have been developed which seek to move beyond the paradigm of purely downgradient tracer transport, and which appeal to the expected turbulent cascades of the large-scale flow. Both observations (Stammer, 1997; Wang et al., 2010; Callies and Ferrari, 2013; Rocha et al., 2016) and modeling studies (Klein et al., 2008; Capet et al., 2008; Sasaki and Klein, 2012; Rocha et al., 2016) indicate that large-scale ocean turbulence is quasi-geostrophic, featuring an upscale cascade of kinetic energy (Scott and Wang, 2005; Scott and Arbic, 2007) and a forward (downscale) cascade of potential enstrophy. The upscale energy cascade has frequently been a focus of recent parameterization methods because it implies an energy flux away from dissipative scales, thereby conflicting with the need to dissipate resolved kinetic energy to maintain numerical stability. This dissipation implies that the upscale cascade may be attenuated or arrested unphysically, and the resulting energy loss may be complicit in causing the large-scale circulation in global models to be weak compared to observations (e.g. Kjellsson and Zanna, 2017).

There is considerable debate about the optimal way to approach the problem of compensating for the unphysical energy loss at large scales, and it is unclear whether using scale- and flow-aware dissipation (e.g. Bachman et al., 2017; Pearson et al., 2017) is sufficient in this regard. Viscous dissipation in these models is both unavoidable and necessary, and mimicking the upscale energy cascade by re-injecting the dissipated energy at larger spatial scales is warranted. Deterministic and stochastic approaches (e.g. Frederiksen and Davies, 1997; Berloff, 2005; Duan and Nadiga, 2007; Kitsios et al., 2013; Grooms and Majda, 2013; Porta Mana and Zanna, 2014; Jansen and Held, 2014; Jansen et al., 2015; Zanna et al., 2017) have shown promise in modeling certain dynamics characteristic of upscale energy transfer, such as upgradient momentum fluxes and energy backscatter. A more recent class of eddy closures (e.g. Porta Mana and Zanna, 2014; Anstey and Zanna, 2017; Zanna et al., 2017), which are based on the idea that turbulent stresses may be modeled by assuming a non-Newtonian stress-strain relation (e.g Ericksen, 1956; Rivlin, 1957; Crow, 1968; Lumley, 1970; Meneveau and Katz, 2000), may also be capable of modeling these dynamics.

Another branch of literature pertaining to the Lagrangian-averaged Navier–Stokes-alpha (LANS-α) model also addresses the issue of correcting the large-scale energy. LANS-α has proven to be a skillful turbulence model in both engineering- (Chen et al., 1998, 1999; Holm

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https://doi.org/10.1016/j.ocemod.2018.04.007
Received 9 November 2017; Received in revised form 20 April 2018; Accepted 25 April 2018
Available online 27 April 2018
1463-5003/

et al., 2005) and geophysical-scale flows (Holm and Nadiga, 2003; Hecht et al., 2008a, 2008b; Petersen et al., 2008). The latter experiments showed a remarkable energization of the eddy and mean kinetic energy fields equivalent to doubling the model resolution (Petersen et al., 2008), thereby achieving a similar large-scale energy amplification similar to the newer closures mentioned above. For geophysical flows, this energization occurs because LANS-α effectively shifts the kinetic energy injection scale to lower wavenumbers, helping to spur the onset of baroclinic instability (Holm and Wingate, 2005). As such, this aspect of LANS-α is especially exciting for eddy-permitting ocean modeling.

An interesting property of the LANS-α model is that its governing equations (Foias et al., 2001) are actually variants of the equations for an incompressible, homogeneous fluid of second grade (Dunn and Fosdick, 1974; Dunn and Rajagopal, 1995), whose stress-strain relation was recently examined in detail by Anstey and Zanna (2017, hereafter AZ) as a candidate turbulence closure for large-scale ocean models. It can be shown that if the AZ closure is extended to include the “memory” term neglected in their analysis, one recovers a set of governing equations which exhibit similarities to those used in LANS-α. For the ocean modeler, the mathematical similarity between AZ and LANS-α is obscured by the vastly different notation used in their respective analyses, and the fact that the literature dealing with non-Newtonian fluids is often outside the scope of oceanographic research. That the governing equations of these models have similarities is significant, for one might expect that key advantages of one model, such as the baroclinic energization by LANS-α mentioned above, would thus be conferred by AZ as well; while the eddy geometry from AZ, based on (e.g. Waterman and Lilly, 2015), can be translated to the LANS-α model.

The purpose of this paper is to show the similarities and discrepancies between the LANS-α model, AZ, and second-grade fluid equations using tensor notation as in AZ. The connection between LANS-α and second-grade fluids has been mentioned in previous literature (e.g. Foias et al., 2001; Marsden and Shkoller, 2001, among others) but to the authors’ knowledge was never derived explicitly. Here, by starting from the stress tensor for a second-grade fluid, we derive and explicitly show how the connection arises between the LANS-α model, the Rivlin–Ericksen stress and the AZ closure, while also providing a synthesis of previous ideas. To allow a thorough comparison of the different turbulence models, we will extend the original AZ formulation to 3D and show the equations that would result if one were to follow their approach and break the second Rivlin–Ericksen stress tensor into “memory” and “deformation” parts. For brevity the Coriolis and external body forces are left out of these derivations, although they can be added back in without affecting any part of the analysis.

2. A brief discussion of second-grade fluids and LANS-α

An extensive body of literature exists which discusses the mathematics and physics of both non-Newtonian fluids and LANS-α, whose scope deserves a far more thorough discussion than is possible here.[1] Here only a few key elements in the development of both are mentioned.

Much of the nomenclature used in discussing non-Newtonian fluids stems from continuum mechanics, and is intended to extend to general coordinate systems and moving frames of reference. Objects defined below which may have familiar names in the oceanographic literature, such as the *strain rate tensor*, **S**, or *vorticity tensor*, **W**, may instead be formally referred to as the *rate of deformation tensor* and *spin tensor*, respectively. Other functions of these tensors and their time derivatives often appear. To keep this derivation accessible, here we will restrict consideration to a Cartesian, Eulerian frame, with velocity vector $\mathbf{u} = (u, v, w)$. The velocity gradient tensor is defined as

$$\nabla\mathbf{u} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}, \tag{1}$$

and its symmetric and antisymmetric parts as,

$$\mathbf{S} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \qquad \text{and} \qquad \mathbf{W} = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T), \tag{2}$$

where $\nabla\mathbf{u}^T$ refers to the transpose of (1). Additionally, we will assume the fluid is Boussinesq, allowing us to replace variable density, ρ, with a constant, ρ_0.

An incompressible second-grade fluid is a particular class of non-Newtonian *Rivlin–Ericksen fluids of differential type* (Rivlin and Ericksen, 1955), which are materials in which only a very short part of the deformation history has an influence on the stress. Mathematically, this simply means that the stress in Rivlin–Ericksen fluids is treated as a function of the velocity gradient and some number of its higher time derivatives. For a second-grade fluid, the stress tensor is the sum of all tensors which can be formed using up to two spatial derivatives of the velocity field, and can be written (Criminale et al., 1958; Coleman and Noll, 1960)

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{3}$$

Here p is the thermodynamic pressure and μ, α_1 and α_2 are material moduli and are properties of the flow rheology, with μ being the familiar dynamic viscosity. While cases where the moduli are treated as functions of the strain rate have been considered (e.g. Criminale et al., 1958), the rheology is generally assumed to be homogeneous so that the viscosity and other stress moduli are treated as constants. $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second Rivlin–Ericksen tensors, which represent the lowest-order approximations of the deformation history:

$$\mathbf{A}_1 = 2\mathbf{S} \tag{4}$$

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \nabla\mathbf{u}^T\mathbf{A}_1 + \mathbf{A}_1\nabla\mathbf{u}. \tag{5}$$

The operator

$$D/Dt = \partial_t + \mathbf{u}\cdot\nabla \tag{6}$$

is the usual material derivative. The equations of motion for this system state that the acceleration of the fluid is equal to the divergence of the stress tensor,

$$\frac{D\mathbf{u}}{Dt} = \frac{1}{\rho_0}\mathrm{div}\,\boldsymbol{\sigma}. \tag{7}$$

Together with the additional thermodynamical constraints (Dunn and Fosdick, 1974)

$$\mu \geq 0, \qquad \alpha_1 + \alpha_2 = 0, \qquad \alpha_1 \geq 0, \tag{8}$$

we will show that the momentum equations for second-grade fluids take the general form

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + \nabla\mathbf{u}^T\cdot\mathbf{v} &= -\frac{1}{\rho_0}\nabla P + \nu\nabla^2\mathbf{u} + \mathscr{F} \\ \mathbf{v} &= (1 - \alpha\nabla^2)\mathbf{u} \\ \alpha &= \frac{\alpha_1}{\rho_0}. \end{aligned} \tag{9}$$

The Lagrangian derivative in (9), and in all subsequent expressions, remains as defined in (6). Here we have introduced the kinematic viscosity, $\nu = \mu/\rho_0$, and a rescaled stress modulus, α, for brevity and assume that they are both constant and positive. P is a modified pressure whose exact form depends on whether one chooses to neglect terms in the nonlinear stress $\mathbf{A}_2$, which is the scenario explored by AZ. $\mathscr{F}$ represents extra terms that also appear in the momentum equations

[1] For an excellent retrospective on the theory of incompressible second-grade fluids, the reader is encouraged to consult Dunn and Rajagopal (1995). Likewise, an interesting exposition on the development of LANS-α from concept to turbulence closure can be found in Holm et al. (2005).