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A fully stochastic approach to limit theorems for iterates of Bernstein operators

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Abstract

This paper presents a stochastic approach to theorems concerning the behavior of iterations of the Bernstein operator B_n taking a continuous function $f \in C[0, 1]$ to a degree-n polynomial when the number of iterations k tends to infinity and n is kept fixed or when n tends to infinity as well. In the first instance, the underlying stochastic process is the so-called Wright–Fisher model, whereas, in the second instance, the underlying stochastic process is the Wright–Fisher diffusion. Both processes are probably the most basic ones in mathematical genetics. By using Markov chain theory and stochastic compositions, we explain probabilistically a theorem due to Kelisky and Rivlin, and by using stochastic calculus we compute a formula for the application of B_n a number of times $k = k(n)$ to a polynomial f when $k(n)/n$ tends to a constant.

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1. Introduction

About 100 years ago, Bernstein [3] introduced a sequence of polynomials approximating a continuous function on a compact interval. That polynomials are dense in the set of continuous functions was shown by Weierstrass [25], but Bernstein was the first to give a concrete method, one that has withstood the test of time. We refer to [19] for a history of approximation theory, including *inter alia* historical references to Weierstrass' life and work and to the subsequent work of Bernstein. Bernstein's approach was probabilistic and is nowadays included in numerous textbooks on probability theory, see, e.g., [20, p. 54] or [4, Theorem 6.2].

Several years after Bernstein's work, the nowadays known as Wright–Fisher stochastic model was introduced and proved to be a founding one for the area of mathematical genetics. The work was done in the context of Mendelian genetics by Ronald A. Fisher [10,11] and Sewall Wright [26].

This paper aims to explain the relation between the Wright–Fisher model and the Bernstein operator B_n, that takes a function $f \in C[0,1]$ and outputs a degree-n approximating polynomial. Bernstein's original proof was probabilistic. It is thus natural to expect that subsequent properties of B_n can also be explained via probability theory. In doing so, we shed new light to what happens when we apply the Bernstein operator B_n a large number of times k to a function f. In fact, things become particularly interesting when k and n converge simultaneously to ∞. This convergence can be explained by means of the original Wright–Fisher model as well as a continuous-time approximation to it known as Wright–Fisher diffusion.

Our paper was inspired by the Monthly paper of Abel and Ivan [1] that gives a short proof of the Kelisky and Rivlin theorem [16] regarding the limit of the iterates of B_n when n is fixed. We asked what is the underlying stochastic phenomenon that explains this convergence and found that it is the composition of independent copies of the empirical distribution function of n i.i.d. uniform random variables. The composition turns out to be precisely the Wright–Fisher model. Being a Markov chain with absorbing states, 0 and 1, its distributional limit is a random variable that takes values in $\{0, 1\}$; whence the Kelisky and Rivlin theorem [16].

Composing stochastic processes of different type (independent Brownian motions) has received attention recently, e.g., in [6]. Indeed, such compositions often turn out to have interesting, nontrivial, limits [5]. Stochastic compositions become particularly interesting when they explain some natural mathematical or physical principles. This is what we do, in a particular case, in this paper. Besides giving fresh proofs to some phenomena, stochastic compositions help find what questions to ask as well.

We will specifically provide probabilistic proofs for a number of results associated to the Bernstein operator (1). First, we briefly recall Bernstein's probabilistic proof (Theorem 1) that says that $B_n f$ converges uniformly to f as the degree n converges to infinity. Second, we look at iterates B_n^k of B_n, meaning that we compose B_n k times with itself and give a probabilistic proof of the Kelisky and Rivlin theorem stating that $B_n^k f$ converges to $B_1 f$ as the number of iterations k tends to infinity (Theorem 2). Third, we exhibit, probabilistically, a geometric rate of convergence to the Kelisky and Rivlin theorem (Proposition 1). Fourth, we examine the limit of $B_n^k f$ when both n and

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