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Gelfand–Mazur Theorems in normed algebras: A survey

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Abstract

The Gelfand–Mazur Theorem, a very basic theorem in the theory of Banach algebras states that: (Real version) Every real normed division algebra is isomorphic to the algebra of all real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$; (Complex version) Every complex normed division algebra is isometrically isomorphic to $\mathbb{C}$. This theorem has undergone a large number of generalizations. We present a survey of these generalizations and also discuss some closely related unsettled issues.

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1. The Gelfand–Mazur theorem

An *algebra* $\mathcal{A}$ over a field F is a vector space over F which is also a ring such that for all $x, y \in \mathcal{A}$ and for all $\lambda \in F$, $\lambda(xy) = x(\lambda y) = (\lambda x)y$ holds. We assume $\mathcal{A}$ to be associative and not necessarily having identity element. We shall take F to be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, and accordingly call $\mathcal{A}$ to be a *real algebra* or a *complex algebra*. A *divisionalgebra* is an algebra with identity such that every non zero element is invertible. A *normed algebra* $(\mathcal{A}, \|.\|)$ is an algebra $\mathcal{A}$ together with a

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norm $\|.\|$ such that $(\mathcal{A}, \|.\|)$ is a normed linear space and the norm is submultiplicative, that is, $\|xy\| \leq \|x\|\|y\|$ for all x, y in $\mathcal{A}$. A *Banach algebra* is a normed algebra that is a Banach space. Banach algebras exhibit a fruitful interplay between Algebra and Analysis resulting into a rich theory of algebras in analysis [11,15,30,39,41]. The subject has a rich collection of examples from Function Theory, Harmonic Analysis and Linear Operator Theory in Banach and Hilbert Spaces. It has also provided a basic framework for the development of C^*-algebras and von Neumann algebras creating a foundation for the development of noncommutative mathematics of analysis like Noncommutative Probability and Noncommutative Geometry. The following fundamental theorem is a corner stone of Banach Algebras; and it compares in simplicity and beauty with the Liouville Theorem of Complex Analysis. We recall two popular versions of the theorem.

Theorem 1.1 (*Real Version*)**.** *Every real normed division algebra is isomorphic to the set of all real numbers* $\mathbb{R}$*, the complex numbers* $\mathbb{C}$ *or the quaternions* $\mathbb{H}$*.*

Theorem 1.2 (*Complex Version*)**.** *Every complex normed division algebra is isometrically isomorphic to* $\mathbb{C}$*.*

The division algebra $\mathbb{H}$ of quaternions is the algebra consisting of elements of form $x = \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ subject to the multiplication $ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = -1$, 1 being the multiplicative identity. The theorem is a natural sequel to the classical Frobenius Theorem [17,20] that states that a real finite dimensional division algebra is isomorphic to $\mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$; and it illustrates the power of methods of Analysis to study infinite dimensional algebras. This is also illustrated by the fact that in a Banach algebra, if an element x is invertible, then all y in an appropriate neighbourhood of x are also invertible. Immediately after the appearance of first papers in Banach algebras [37,49,50], Mazur [36] announced the theorem without proof. It is stated by some authors that Mazur's original submission contained a proof. But it was deleted from the final paper due to Editor's insistence on shortening the proof. A very elegant proof of the complex version, based on the Liouville Theorem for entire functions was given by Gelfand in his famous paper [22]. Mazur's original proof based incidentally on Liouville Theorem for harmonic functions became available much later in a book by Zelazko [53]. It can also be found in [34]. Thus chronologically the theorem deserves to be called the Mazur–Gelfand Theorem; but the term Gelfand–Mazur Theorem has become very popular and well established by now. Like some other fundamental theorems, Gelfand–Mazur Theorem and its avatars have also inspired elementary proofs thereof [14,21,29,33,40,41,45].

Let $\mathcal{A}$ be a complex normed algebra with identity 1. A proof due to Arens [2] of the complex version uses the Liouville Theorem. A major step in this proof is to prove that for $x \in \mathcal{A}$ the resolvent function $R_x(\lambda) := (\lambda 1 - x)^{-1}$, $\lambda \in \mathbb{C}$ is analytic wherever it is defined. A consequence of the theorem is the most fundamental result of Banach Algebras that for each $x \in \mathcal{A}$, the *spectrum* $sp(x) := \{\lambda \in \mathbb{C} : (\lambda 1 - x)$ is not invertible in $\mathcal{A}\}$ is non empty and compact. On the other hand, Gelfand's proof as well as the elementary proofs due to Kametani [31] and Rickart [40,41] establish first the non emptiness of spectra from which the theorem follows easily. (The elementary proof due to Kametani and Rickart is based on decomposing the polynomial $x^n - 1$ in terms of linear factors