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ON THE PERFECTION OF SCHEMES

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Abstract. We present a detailed and elementary construction of the inverse perfection of a scheme and discuss some of its main properties. We also establish a number of auxiliary results (for example, on inverse limits of schemes) which do not seem to appear in the literature.

1. Introduction

Let p be a prime number and let $\mathbb{F}_p$ denote the field with p elements. In the course of our review of the construction of the perfect Greenberg functor in [BGA], we were hampered by the lack of an adequate reference work on the subject of (inverse) perfections of $\mathbb{F}_p$-schemes. Although the classical reference [Gre] presents in some detail the construction of the inverse perfection Y^{pf} of an $\mathbb{F}_p$-scheme Y, it does not discuss its main properties. On the other hand, the relatively recent preprint [BS] does briefly discuss some of the main aspects of the indicated construction (see [BS, Lemmas 3.4 and 3.8], parts of which overlap with some of the results presented here), but it does not address properties of the perfection of $\mathbb{F}_p$-group schemes, as such properties are not relevant in [BS]. Our aim in this paper is to present a detailed and elementary construction of the inverse perfection of an $\mathbb{F}_p$-scheme and discuss some of its properties. The (inverse) perfection functor has played, a continues to play, a significant role in algebraic geometry (see, for example, [Ser1, Ser2, BD, BW, Pep, KL]). We believe that our presentation will be useful to all students and researchers that at some point in their studies will need to consider the (inverse) perfection of an $\mathbb{F}_p$-scheme.

We briefly indicate the contents of the individual Sections.

Section 2 presents some basic results on the fpqc and fppf topologies. These statements may be well-known to some readers but, to our knowledge, they do not appear in the literature. Section 3 discusses certain basic properties of projective limits of schemes that, surprisingly, we could not find in the standard literature on the subject. In particular, Proposition 3.8 shows that, if k is any field, then the inverse limit functor is exact on certain types of "Mittag-Leffler" short exact sequences of projective systems *in the category of commutative k-group schemes*. Section 4 is a detailed discussion of the construction of the *perfect closure* (or *direct*

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