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The Balian–Low theorem and noncommutative tori

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Abstract

We point out a link between the theorem of Balian and Low on the non-existence of well-localized Gabor–Riesz bases and a constant curvature connection on projective modules over noncommutative tori.

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1. Introduction

The theorem of Balian–Low on the non-existence of well-localized Gabor–Riesz bases for $L^2(\mathbb{R})$ is one of the cornerstones of time-frequency analysis [2,15]. For the formulation we first introduce the multiplication operator and differentiation operator, denoted by $(\nabla_1 g)(t) = 2\pi \mathrm{i}\, t\, g(t)$ and $(\nabla_2 g)(t) = g'(t)$, respectively. Let $\pi(z)g(t) = e^{2\pi i \omega t} g(t-x)$ be the time-frequency shift of a function g by $z = (x, \omega)$ in phase space. Gabor studied in [8] systems of the form $\mathcal{G}(g, \theta\mathbb{Z} \times \mathbb{Z}) = \{\pi(\theta k, l)g : k, l \in \mathbb{Z}\}$, so-called *Gabor systems* with *Gabor atom* g. The density theorem for Gabor frames says that if $\mathcal{G}(g, \theta\mathbb{Z} \times \mathbb{Z})$ θ is a frame, then $\theta \in (0, 1]$.

A natural question about Gabor systems $\mathcal{G}(g, \theta\mathbb{Z} \times \mathbb{Z})$ for a fixed Gabor atom g is to study for which θ the system $\mathcal{G}(g, \theta\mathbb{Z} \times \mathbb{Z})$ is a frame. If g is well-localized in time and

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frequency, then $\theta = 1$ will be excluded as for the Gaussian. It turns out that for some Gabor atoms g, such as the Gaussian or any totally positive function of finite type $\mathcal{G}(g, \theta\mathbb{Z} \times \mathbb{Z})$ is a Gabor frame for any θ in $(0, 1)$, [13,18,24]. On the other hand the answer for the indicator function of an interval $[0, c]$ is much more intricate and the values θ and c for which one gets a Gabor frame are known as Janssen's tie [7,14]. The theorem of Balian–Low provides an explanation for these facts.

Theorem 1.1 (*Balian–Low*)**.** *Suppose the Gabor system $\mathcal{G}(g, \mathbb{Z}^2)$ is an orthonormal basis for $L^2(\mathbb{R})$. Then*

$$\Big(\int_{\mathbb{R}} |(\nabla_1 g)(t)|^2\, dt\Big)\Big(\int_{\mathbb{R}} |(\nabla_2 g)(t)|^2\, dt\Big) = \infty.$$

In particular, the theorem of Balian–Low implies that if $\mathcal{G}(g, \mathbb{Z}^2)$ is an orthonormal basis for $L^2(\mathbb{R})$, then g is not well-localized in time and frequency, e.g. g cannot be in the Schwartz class $\mathscr{S}(\mathbb{R})$ or in Feichtinger's algebra $S_0(\mathbb{R})$. The definition of Gabor systems does not indicate a link to regularity properties of the Gabor atom. Hence, the very reason for the incompatibility between orthonormal Gabor bases of the form $\mathcal{G}(g, \mathbb{Z}^2)$ and good time-frequency localization is not well understood despite the vast literature on the Balian–Low theorem [1–4,9,10,12,19,20]. Note that some authors refer to statements of the form Theorem 1.1 as *weak* Balian–Low theorems.

The main aim of this investigation is to present an approach to Gabor frames that provides an explanation of the link between regularity properties of Gabor atoms and their behavior at the critical density. We are building on the correspondence between Gabor frames and projective modules over noncommutative tori [16,17]. The standard argument to demonstrate Theorem 1.1 is due to Battle [3]. We are demonstrating that Battle's argument is best understood in terms of noncommutative geometry.

2. Noncommutative tori

In noncommutative geometry one attempts to define geometric objects and notions for general C^*-algebras. For our purpose we need the noncommutative torus $\mathcal{A}_\theta$ equipped with its structure as a noncommutative manifold. We briefly recall the construction of vector bundles over noncommutative tori, which are finitely generated projective modules over $\mathcal{A}_\theta$, the differential structure on $\mathcal{A}_\theta$ is given by derivations and on the vector bundles by a connection, and the notion of curvature of a connection [5].

We denote the operators $\pi(0, 1)$ and $\pi(\theta, 0)$ as M_1 and T_θ, respectively. Note that we have $M_1 T_\theta = e^{2\pi i\theta} T_\theta M_1$ and hence the norm closure of $\{\pi(k\theta, l) : k, l \in \mathbb{Z}\}$ defines the noncommutative torus $\mathcal{A}_\theta$, [21]. The smooth noncommutative torus is the subalgebra $\mathcal{A}_\theta^\infty$ of $\mathcal{A}_\theta$ consisting of operators

$$\pi(\mathbf{a}) = \sum_{k,l\in\mathbb{Z}} a_{kl}\pi(\theta k, l), \qquad \text{for} \quad \mathbf{a} = (a_{kl}) \in \mathscr{S}(\mathbb{Z}^2). \tag{1}$$

The standard derivations on $\mathcal{A}_\theta$ are given by

$$\partial_1(a) = 2\pi \mathrm{i} \sum_{k,l} k a_{kl}\pi(\theta k, l) \quad \text{and} \qquad \partial_2(a) = 2\pi \mathrm{i} \sum_{k,l} l a_{kl}\pi(\theta k, l)\,.$$

The Schwartz space $\mathscr{S}(\mathbb{R})$ turns out to be vector bundle over $\mathcal{A}_\theta^\infty$, [5,16,22].