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On the constants for some fractional Gagliardo–Nirenberg and Sobolev inequalities

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Abstract

We consider the inequalities of Gagliardo–Nirenberg and Sobolev in $\mathbb{R}^d$, formulated in terms of the Laplacian Δ and of the fractional powers $D^n := \sqrt{-\Delta}^{\,n}$ with real $n \geqslant 0$; we review known facts and present novel, complementary results in this area. After illustrating the equivalence between these two inequalities and the relations between the corresponding sharp constants and maximizers, we focus the attention on the $\mathcal{L}^2$ case where, for all sufficiently regular $f : \mathbb{R}^d \to \mathbb{C}$, the norm $\|D^j f\|_{\mathcal{L}^r}$ is bounded in terms of $\|f\|_{\mathcal{L}^2}$ and $\|D^n f\|_{\mathcal{L}^2}$, for $1/r = 1/2 - (\vartheta n - j)/d$, and suitable values of j, n, ϑ (with j, n possibly noninteger). In the special cases $\vartheta = 1$ and $\vartheta = j/n + d/2n$ (i.e., $r = +\infty$), related to previous results of Lieb and Ilyin, the sharp constants and the maximizers can be found explicitly; we point out that the maximizers can be expressed in terms of hypergeometric, Fox and Meijer functions. For the general $\mathcal{L}^2$ case, we present two kinds of upper bounds on the sharp constants: the first kind is suggested by the literature, the second one is an alternative proposal of ours, often more precise than the first one. We also derive two kinds of lower bounds. Combining all the available upper and lower bounds, the sharp constants are confined to quite narrow intervals. Several examples are given, including the numerical values of the previously mentioned bounds.

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1. Introduction

In this paper we work in $\mathbb{R}^d$, using the Laplacian Δ, the operator $D := \sqrt{-\Delta}$ and its powers D^n with real exponent $n \geqslant 0$. In the sequel f stands for a complex-valued function on $\mathbb{R}^d$, with suitable regularity properties.

We consider the embedding inequalities of Gagliardo [11], Nirenberg [27] and Sobolev [32]. The terms *Gagliardo–Nirenberg inequality* and *Sobolev inequality* are used to indicate, respectively, the inequalities[1]

$$\|D^j f\|_{\mathcal{L}^r} \leqslant G\, \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^{\vartheta} \qquad \Big(\frac{1}{r} = \frac{1-\vartheta}{p} + \frac{\vartheta}{q} - \frac{\vartheta n - j}{d}\Big), \tag{1.1}$$

$$\|D^j f\|_{\mathcal{L}^r} \leqslant S\, (\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t} \qquad (r \text{ as in } (1.1)), \tag{1.2}$$

holding if the parameters p, q, j, n, ϑ, t fulfill appropriate conditions. Here and in the sequel, $\mathcal{L}^p$ is the usual space $L^p \equiv L^p(\mathbb{R}^d, \mathbb{C})$ for $p \in [1, +\infty)$, while $\mathcal{L}^\infty$ is the subspace of $C(\mathbb{R}^d, \mathbb{C})$ made of the functions vanishing at infinity, with the usual sup norm (see the forthcoming Eq. (2.11), and the related comments).

The inequalities (1.1) (1.2) are found to be equivalent via appropriate scaling considerations. We are especially interested in their $\mathcal{L}^2$ versions which are obtained setting $p = q = t = 2$, and read

$$\|D^j f\|_{\mathcal{L}^r} \leqslant G\, \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \qquad \Big(\frac{1}{r} = \frac{1}{2} - \frac{\vartheta n - j}{d}\Big), \tag{1.3}$$

$$\|D^j f\|_{\mathcal{L}^r} \leqslant S \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \qquad (r \text{ as in } (1.3)). \tag{1.4}$$

They hold under suitable conditions on j, n, ϑ, given in the forthcoming Eq. (5.9) and here anticipated:

$$0 \leqslant \vartheta \leqslant 1\,, \quad 0 \leqslant n, j < +\infty\,, \quad 0 \leqslant \vartheta n - j \leqslant \frac{d}{2}\,, \quad \vartheta \neq 1 \text{ if } n = j + \frac{d}{2}\,; \tag{1.5}$$

we write $G(j, n, \vartheta)$ and $S(j, n, \vartheta)$ for the sharp constants of (1.3) and (1.4), respectively. The aims of this paper are as follows.

[1] The association of the cited authors to either (1.1) or (1.2) is to some extent conventional; in particular, the cited papers of Gagliardo consider mainly the inequality (1.2). However, these historical aspects are not relevant for our purposes.