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Weakly hyperbolic involutions

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Received 23 June 2016; received in revised form 22 June 2017

Abstract

Pfister's Local–Global Principle states that a quadratic form over a (formally) real field is weakly hyperbolic (i.e. represents a torsion element in the Witt ring) if and only if its total signature is zero. This result extends naturally to the setting of central simple algebras with involution. The present article provides a new proof of this result and extends it to the case of signatures at preorderings. Furthermore the quantitative relation between nilpotence and torsion is explored for quadratic forms as well as for central simple algebras with involution.

MSC 2010: primary 11E10; secondary 11E04; 11E81; 12D15; 16K20; 16W10

Keywords: Real field; Quadratic form; Signature; Local–global; Quaternion algebra; Witt group

1. Introduction

Pfister's Local–Global Principle says that a regular quadratic form over a (formally) real field represents a torsion element in the Witt ring if and only if its signature at each ordering of the field is zero. This result has been extended in [13] to central simple algebras with involution.

The theory of central simple algebras with involution is a natural extension of quadratic form theory. On the one hand many concepts and results from quadratic form theory have

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http://dx.doi.org/10.1016/j.exmath.2017.08.006

been extended to algebras with involution. On the other hand quadratic forms are used as tools in the study of algebras with involution. Examples include involution trace forms and spaces of similitudes.

In this article we are interested in weakly hyperbolic algebras with involution, a natural generalization of torsion quadratic forms, which was considered first in [20, Chap. 5]. In [13] such algebras with involution were characterized as those having trivial signature at all orderings of the base field, thus generalizing Pfister's Local–Global Principle.

We give a new exposition of this result, highlighting several new aspects, and obtain some extensions. In particular, we provide bounds for the torsion order of nilpotent quadratic forms and extend this result to involutions. We attempt to minimize the use of hermitian forms and treat algebras with involution as direct analogues of quadratic forms. We only consider fields of characteristic different from 2 since the notion of weak hyperbolicity is only interesting in this case.

The structure of this article is as follows. In Section 2 we recall the necessary background material from the theory of quadratic forms and ordered fields as well as Pfister's Local–Global Principle for quadratic forms in a generalized version, relative to preorderings. We also touch on the quantitative aspect of the relation between nilpotence and torsion, using Lewis' annihilating polynomials.

In Section 3 we recall the basic terminology for algebras with involution, consider their relations to quaternion algebras and quadratic forms and study involution trace forms.

In Section 4 we treat the notion of hyperbolicity for algebras with involution and cite the relevant results about its behaviour under field extensions.

In Section 5 we turn to the study of algebras with involution over real fields. In Theorem 5.2 we obtain a classification over real closed fields. We then provide a uniform definition of signatures for involutions of both kinds with respect to an ordering. Signatures were introduced in [12] for involutions of the first kind and in [16] for involutions of the second kind, and both cases are treated in [7, (11.10), (11.25)].

In Section 6 we give a new proof of the main result of [13], an analogue of Pfister's Local–Global Principle for algebras with involution (Theorem 6.5). In Theorem 6.7 we extend this result to a local–global principle for T-hyperbolicity with respect to a preordering T.

In its original version for quadratic forms as well as in the generalized version for algebras with involution Pfister's Local–Global Principle relates the hyperbolicity of tensor powers to the hyperbolicity of multiples. For quadratic forms this corresponds to the relation between nilpotence and torsion for an element of the Witt ring. In Section 7 we consider the quantitative aspect of this relation in the setting of algebras with involution.

Some of the essential ideas contained in Sections 5 and 6 germinated in the MSc thesis of Beatrix Bernauer [2], prepared under the guidance of the first named author.

2. Pfister's Local–Global Principle

We refer to [9] and [18] for the foundations of quadratic form theory over fields and the relevant terminology. Let K be a field of characteristic different from 2. We denote by $K^\times$ the multiplicative group of K, by $K^{\times 2}$ the subgroup of nonzero squares, and by

Please cite this article in press as: K.J. Becher, T. Unger, Weakly hyperbolic involutions, Expo. Math. (2017), http://dx.doi.org/10.1016/j.exmath.2017.08.006.