Finite Fields and Their Applications 54 (2018) 133–150

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

New maximum scattered linear sets of the projective line ☆

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A R T I C L E I N F O

Article history:
Received 28 July 2017
Received in revised form 23 July 2018
Accepted 3 August 2018
Available online xxxx
Communicated by James W.P. Hirschfeld

MSC:
51E20
51E22
05B25

Keywords:
Linear set
Scattered subspace
MRD-code

A B S T R A C T

In [2] and [18] are presented the first two families of maximum scattered $\mathbb{F}_q$-linear sets of the projective line $\mathrm{PG}(1,q^n)$. More recently in [22] and in [5], new examples of maximum scattered $\mathbb{F}_q$-subspaces of $V(2,q^n)$ have been constructed, but the equivalence problem of the corresponding linear sets is left open.
Here we show that the $\mathbb{F}_q$-linear sets presented in [22] and in [5], for $n=6,8$, are new. Also, for q odd, $q \equiv \pm 1,\, 0 \pmod 5$, we present new examples of maximum scattered $\mathbb{F}_q$-linear sets in $\mathrm{PG}(1,q^6)$, arising from trinomial polynomials, which define new $\mathbb{F}_q$-linear MRD-codes of $\mathbb{F}_q^{6\times 6}$ with dimension 12, minimum distance 5 and left idealiser isomorphic to $\mathbb{F}_{q^6}$.

☆ The research was supported by Ministry for Education, University and Research of Italy MIUR (Project PRIN 2012 "Geometrie di Galois e strutture di incidenza") and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA–INdAM). The first author is supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The first author acknowledges the support of OTKA Grant No. K 124950.
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https://doi.org/10.1016/j.ffa.2018.08.001
1071-5797/

1. Introduction

Linear sets are natural generalisations of subgeometries. Let $\Lambda = \mathrm{PG}(W, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$, where W is a vector space of dimension r over $\mathbb{F}_{q^n}$. A point set L of Λ is said to be an $\mathbb{F}_q$-*linear set* of Λ of rank k if it is defined by the non-zero vectors of a k-dimensional $\mathbb{F}_q$-vector subspace U of W; that is,

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

The maximum field of linearity of an $\mathbb{F}_q$-linear set L_U is $\mathbb{F}_{q^t}$ if $t \mid n$ is the largest integer such that L_U is an $\mathbb{F}_{q^t}$-linear set. Two linear sets L_U and L_W of $\mathrm{PG}(r-1, q^n)$ are said to be $\mathrm{P\Gamma L}(r, q^n)$-*equivalent* if there is an element ϕ in $\mathrm{P\Gamma L}(r, q^n)$ such that $L_U^{\phi} = L_W$. It may happen that two $\mathbb{F}_q$-linear sets L_U and L_W of $\mathrm{PG}(r-1, q^n)$ are $\mathrm{P\Gamma L}(r, q^n)$-equivalent even if the two $\mathbb{F}_q$-vector subspaces U and W are in different orbits of $\mathrm{\Gamma L}(r, q^n)$; see [7] and [3] for further details. In recent years, starting from the paper [17] by Lunardon, linear sets have been used to construct or characterise various objects in finite geometry, such as blocking sets and multiple blocking sets in finite projective spaces, two-intersection sets in finite projective spaces, translation spreads of the Cayley Generalized Hexagon, translation ovoids of polar spaces, semifield flocks and finite semifields. For a survey on linear sets we refer the reader to [21]; see also [12]. In applications, it is crucial to have methods to decide whether two linear sets are $\mathrm{P\Gamma L}(r, q^n)$-equivalent or not.

In this paper we focus on *maximum scattered* $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$ with maximum field of linearity $\mathbb{F}_q$, that is, $\mathbb{F}_q$-linear sets of rank n of $\mathrm{PG}(1, q^n)$ of size $(q^n-1)/(q-1)$. If L_U is a maximum scattered $\mathbb{F}_q$-linear set, then U is a *maximum scattered* $\mathbb{F}_q$-subspace.

If the point $\langle (0,1) \rangle_{\mathbb{F}_{q^n}}$ is not contained in the linear set L_U of rank n of $\mathrm{PG}(1, q^n)$ (which we can always assume after a suitable projectivity), then $U = U_f := \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ for some q-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ in $\mathbb{F}_{q^n}[x]$. In this case we will denote the associated linear set by L_f. The known non-equivalent, under $\mathrm{\Gamma L}(2, q^n)$, maximum scattered $\mathbb{F}_q$-subspaces are the following:

(1) $U_s^{1,n} := \{(x, x^{q^s}) : x \in \mathbb{F}_{q^n}\}$, $1 \le s \le n-1$, $\gcd(s, n) = 1$ ([2,8]);
(2) $U_{s,\delta}^{2,n} := \{(x, \delta x^{q^s} + x^{q^{n-s}}) : x \in \mathbb{F}_{q^n}\}$, $n \ge 4$, $\mathrm{N}_{q^n/q}(\delta) \notin \{0, 1\}$, $q \ne 2$, $\gcd(s, n) = 1$ ([18] for $s = 1$, [22,19] for $s \ne 1$);
(3) $U_{s,\delta}^{3,n} := \{(x, \delta x^{q^s} + x^{q^{s+n/2}}) : x \in \mathbb{F}_{q^n}\}$, $n \in \{6, 8\}$, $\gcd(s, n/2) = 1$, $\mathrm{N}_{q^n/q^{n/2}}(\delta) \notin \{0, 1\}$.

In (3), for the precise conditions on δ and q, see [5, Theorems 7.1 and 7.2]. Also here $q > 2$, otherwise $L_{s,\delta}^{3,n}$ is not scattered.

The stabilisers of the $\mathbb{F}_q$-subspaces above in the group $\mathrm{GL}(2, q^n)$ were determined in [5, Sections 5 and 6]. They have the following orders: