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Non-commutative association schemes and their fusion association schemes

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We give a sufficient condition for a non-commutative association scheme to have a fusion association scheme, and construct non-commutative association schemes from symmetric balanced generalized weighing matrices and generalized Hadamard matrices. We then apply the criterion to these non-commutative association schemes to obtain symmetric fusion association schemes.

1. Introduction

Association schemes are considered as an abstraction of the centralizer of transitive permutation groups, and can be described as the subalgebra of the matrix algebra gen-

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erated by the disjoint $(0, 1)$-matrices which are closed under the transposition and their sum equals to the all-ones matrix [1], [12]. Much of interest is focused on the case of multiplicity free transitive permutation groups. In such cases, the corresponding association schemes are commutative. In the present paper, we consider non-commutative association schemes obtained from some combinatorial objects such as symmetric balanced generalized weighing matrices and generalized Hadamard matrices.

Kharaghani and Torabi [9] showed that for any prime power q, the edge set of complete graph $K_{q^3+q^2+q+1}$ is decomposed into $q+1$ strongly regular graphs sharing q^2+1 disjoint cliques. The decomposition is based on symmetric balanced generalized weighing matrices $BGW(q^2+1, q^2, q^2-1)$ with zero diagonal entries over a cyclic group of order $q+1$, see [3] for details. Motivated by this decomposition, Klin, Reichard and Woldar [10] defined the concept of Siamese objects as a partition of the edge set of the complete graph, and studied it from the view point of graph theory and group theory. In particular, it was shown that an action of $PGL(2, q)$ yields a non-commutative association scheme.

In this paper it is shown that non-commutative association schemes are obtained from the following objects:

- any symmetric balanced generalized weighing matrix $BGW(n+1, n, n-1)$ with zero diagonal entries over a cyclic group C_m of order m,
- a generalized Hadamard matrix attached to finite fields.

Our first example of non-commutative association scheme is obtained from the $BGW(n+1, n, n-1)$ over the cyclic group C_m with $(n, m) = (q^2, q+1)$, includes Kharaghani and Torabi's work, and has the parameters of the non-commutative association scheme obtained by Klin, Reichard and Woldar. We also establish a sufficient condition for a non-commutative association scheme to possess a fusion association scheme, and thus obtain an analog of a part of result by Bannai [1] and Muzychuk [11]. By applying this criterion to our non-commutative association scheme we obtain some symmetric fusion association schemes. Finally, the Wedderburn decomposition (or character table) and the eigenmatrices of the association schemes are explicitly determined.

2. Preliminaries

Throughout this paper, I_n, J_n denote the identity matrix of order n, the all-ones matrix of order n respectively.

Let d be a positive integer. Let X be a finite set of size v and R_i $(i \in \{0, 1, \dots, d\})$ be a nonempty subset of $X \times X$. The *adjacency matrix* A_i of the graph with vertex set X and edge set R_i is a $v \times v$ $(0, 1)$-matrix with rows and columns indexed by X such that $(A_i)_{xy} = 1$ if $(x, y) \in R_i$ and $(A_i)_{xy} = 0$ otherwise. An *association scheme* of d-class is a pair $(X, \{R_i\}_{i=0}^d)$ satisfying the following: