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Plane sections of Fermat surfaces over finite fields

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In this paper, we characterize all curves over $\mathbb{F}_q$ arising from a plane section

$$\mathcal{P} : X_3 - e_0X_0 - e_1X_1 - e_2X_2 = 0$$

of the Fermat surface

$$\mathcal{S} : X_0^d + X_1^d + X_2^d + X_3^d = 0,$$

where $q = p^h = 2d+1$ is a prime power, $p > 3$, and $e_0, e_1, e_2 \in \mathbb{F}_q$. In particular, we prove that any nonlinear component $\mathcal{G} \subseteq \mathcal{P} \cap \mathcal{S}$ is a smooth classical curve of degree $n \leqslant d$ attaining the Stöhr–Voloch bound

$$\#\mathcal{G}(\mathbb{F}_q) \leqslant \frac{1}{2}n(n+q-1) - \frac{1}{2}i(n-2),$$

with $i \in \{0, 1, 2, 3, n, 3n\}$.

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1. Introduction

Let $\mathcal{F}$ be the curve obtained by slicing the Fermat surface

$$\mathcal{S} : X_0^d + X_1^d + X_2^d + X_3^d = 0$$

with the plane

$$\mathcal{P} : X_3 - e_0X_0 - e_1X_1 - e_2X_2 = 0,$$

where d is a positive integer, $e_0, e_1, e_2 \in \mathbb{F}_q$, and $\mathbb{F}_q$ is the finite field with $q = p^h$ elements, with p a prime number. In other words, let

$$\mathcal{F} : X_0^d + X_1^d + X_2^d + (e_0X_0 + e_1X_1 + e_2X_2)^d = 0. \tag{1.1}$$

Characterizing this general curve $\mathcal{F}$ in terms of its rational points and its irreducible and nonsingular components presents many challenges. For instance, the particular case $p = 2$ and $e_0 = e_1 = e_2 = 1$ has been extensively investigated over the past decades (see [4], [8], [9], [11]). In this context, the following result was essential in Hernando and McGuire's proof of an important conjecture regarding exceptional numbers [4].

Theorem *(Hernando–McGuire). The polynomial*

$$\frac{X_0^d + X_1^d + X_2^d + (X_0 + X_1 + X_2)^d}{(X_0 + X_1)(X_0 + X_2)(X_1 + X_2)}$$

has an absolutely irreducible factor defined over $\mathbb{F}_2$ for all d not of the form $d = 2^i + 1$ or $d = 2^{2i} - 2^i + 1$.

In this paper, we consider the problem of studying the curve given in (1.1) from another point of view. Based on techniques developed by Carlin and Voloch [2], we characterize the curve

$$\mathcal{C} : C(X_0, X_1, X_2) = X_0^d + X_1^d + X_2^d + (e_0X_0 + e_1X_1 + e_2X_2)^d = 0, \tag{1.2}$$

where $q = p^h = 2d + 1$ is a prime power, $p > 3$, and e_0, e_1 and e_2 are arbitrary elements in $\mathbb{F}_q$. For such a curve, we give a complete description of the irreducible and nonsingular components and provide their number of $\mathbb{F}_q$-rational points. Consequently, we construct a family of curves attaining the Stöhr–Voloch bound and prove the following theorem, which is the main result of this paper.

Theorem 1.1. *If $\mathcal{C}$ is not the union of d lines, then the following statements hold.*