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On eigenfunctions and maximal cliques of Paley graphs of square order ☆

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In this paper we present a family of maximal cliques of size $\frac{q+1}{2}$ or $\frac{q+3}{2}$, accordingly as $q \equiv 1(4)$ or $q \equiv 3(4)$, in Paley graphs of order q^2, where q is an odd prime power. After that we use the new cliques to define a family of eigenfunctions corresponding to both non-principal eigenvalues and having support size $q+1$, which is the minimum possible value by the weight-distribution bound. Finally, we prove that the constructed eigenfunction comes from an equitable partition.

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Equitable partition

1. Introduction

Let q be an odd prime power, $q \equiv 1(4)$. The *Paley graph* $P(q)$ is the Cayley graph on the additive group $\mathbb{F}_q^+$ of the finite field $\mathbb{F}_q$ with the generating set of all squares in the multiplicative group $\mathbb{F}_q^*$. The Paley graphs $P(q)$ are known to be strongly regular with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$. The well-known Delsarte bound [3] applied to $P(q)$ says that the cardinality of a largest independent set (coclique) is at most $\sqrt{q}$. Since the Paley graphs are self-complementary, the same bound holds for a largest clique of $P(q)$.

The problem of finding the clique (independence) number of Paley graphs is open in general. In [6], the Delsarte bound was improved for infinitely many parameter tuples that correspond to Paley graphs.

In this paper we only consider Paley graphs $P(q^2)$ on the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$, where q is any odd prime power. In this case the subfield $\mathbb{F}_q$ of $\mathbb{F}_{q^2}$ gives a clique of order q, which meets the Delsarte bound. In 1984, Blokhuis [2] determined all cliques and all cocliques of size q in $P(q^2)$ and showed that they are affine images of the subfield $\mathbb{F}_q$. In 1996, maximal cliques of order $\frac{q+1}{2}$ and $\frac{q+3}{2}$ for $q \equiv 1(4)$ and $q \equiv 3(4)$, respectively, were found [1] by Baker et al., but an exhaustive computer search done by them showed that these cliques are not the only cliques of such size. Moreover, there are no known maximal cliques whose size belongs to the gap from $\frac{q+1}{2}$ (from $\frac{q+3}{2}$, respectively) to q. In 2009, Kiermaier and Kurz studied [7] maximal integral point sets in affine planes over finite fields, and found maximal cliques of size $\frac{q+3}{2}$ in $P(q^2)$ for $q \equiv 3(4)$.

Let θ be an eigenvalue of a graph Γ. A real-valued function on the vertex set of Γ is called an *eigenfunction* of the graph Γ corresponding to the eigenvalue θ, if it has at least one non-zero value and for any vertex γ in Γ the condition

$$\theta \cdot f(\gamma) = \sum_{\delta \in \Gamma(\gamma)} f(\delta) \tag{1}$$

holds, where $\Gamma(\gamma)$ is the set of neighbours of the vertex γ. Note that the definition of eigenfunction is equivalent to the definition of eigenvector of the adjacency matrix of Γ corresponding to the eigenvalue θ.

There are several papers devoted to the extremal problem of studying eigenfunctions of graphs with minimum cardinality of support (for more details and motivation, see [8]). In [9], Valyuzhenich found the minimum cardinality of support of an eigenfunction corresponding to the largest non-principal eigenvalue of a Hamming graph $H(n, q)$ and characterised such eigenfunctions with the minimum cardinality of support. In [10],