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On Kassel–Reutenauer q-analog of the sum of divisors and the ring $\mathbb{F}_3[X]/X^2\mathbb{F}_3[X]$

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ABSTRACT

A q-analog $P_n(q)$ of the sum of divisors of n was introduced by C. Kassel and C. Reutenauer in a combinatorial setting and by T. Hausel, E. Letellier, F. Rodriguez-Villegas in a Hodge-theoretic setting. We study the reduction modulo 3 of the polynomial $P_n(q)$ with respect to the ideal $(q^2+q+1)\mathbb{F}_3[q]$.

1. Introduction

Consider the infinite product

$$\theta(w) := (1-w)\prod_{n\geq 1}\frac{(1-q^n w)\left(1-q^n w^{-1}\right)}{(1-q^n)^2}.$$

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The identity

$$\frac{\theta(uv)}{\theta(u)\theta(v)} = \sum_{m,n\geq 0} q^{mn}u^m v^n - \sum_{m,n\geq 1} q^{mn}u^{-m}v^{-n}, \tag{1}$$

is attributed to L. Kronecker[1] [10]. The particular case of (1),

$$\frac{1}{\theta(w)} - \frac{1}{1-w} = \sum_{\substack{n,m\geq 1 \\ n\not\equiv m \pmod 2}} (-1)^n q^{nm/2} w^{(m-n-1)/2}, \tag{2}$$

is attributed to C. Jordan [5, p. 453].

Let $T_n(w) \in \mathbb{Z}\left[w, w^{-1}\right]$ be the coefficient of q^n in the Taylor expansion of (2) at $q = 0$. Let $C_n(q) \in \mathbb{Z}[q]$ be defined by $C_n(q) := (q-1)q^n T_n(q)$. C. Kassel and C. Reutenauer [6,7] proved that, if q is a prime power, then there are precisely $C_n(q)$ ideals I of the group algebra $\mathbb{F}_q\left[\mathbb{Z}^2\right]$ of the free abelian group of rank 2 such that the quotient $\mathbb{F}_q\left[\mathbb{Z}^2\right]/I$ is an n-dimensional vector space over $\mathbb{F}_q$.

T. Hausel, E. Letellier and F. Rodriguez-Villegas [3] proved that $C_n(q)$ is the E-polynomial of the Hilbert scheme $X^{[n]}$ of n points on the algebraic torus $X := \mathbb{C}^\times \times \mathbb{C}^\times$. It is natural to consider the obvious action of the group $\mathbb{C}^\times \times \mathbb{C}^\times$ on the variety X and to extend this action to the punctual Hilbert scheme $X^{[n]}$. Let $\tilde{X}^{[n]} := X^{[n]}//\left(\mathbb{C}^\times \times \mathbb{C}^\times\right)$ be the corresponding GIT-quotient [11]. Denoting $P_n(q) \in \mathbb{Z}[q]$ the E-polynomial of $\tilde{X}^{[n]}$, it follows, using elementary Hodge Theory [4], that $(q-1)^2 P_n(q) = C_n(q)$.

In virtue of (2),

$$P_n(q) = \frac{q^{n-1}}{q-1} \sum_{\substack{d|n \\ d \text{ odd}}} \left(q^{\gamma(d)} - q^{1-\gamma(d)}\right),$$

where $\gamma(d) := \frac{1}{2}\left(\frac{2n}{d} - d + 1\right)$. Using L'Hôpital's rule, it follows that

$$P_n(1) = \lim_{q\to 1} P_n(q) = \sigma(n),$$

where $\sigma(n)$ is the sum of divisors of n. We called [1] $P_n(q)$ the *Kassel–Reutenauer polynomials* because C. Kassel and C. Reutenauer studied some of their number-theoretical properties [6–8]. A more informative name should be *Kassel–Reutenauer q-analog of the sum of divisors.*

It is obvious that $P_n(1)$ is divisible by 3 if $P_n(q)$ belongs to the principal ideal $[3]_q\mathbb{Z}[q]$, where $[3]_q := q^2+q+1$ is the classical q-analog of 3. Nevertheless, the converse statement is not always true. In order to fix this correspondence, we will consider the reduction modulo 3 of the polynomials $P_n(q)$, denoted ${}_3P_n(q)$. The aim of this paper is to prove the following result.

[1] Kronecker's original identity is rather different, but it can be transformed into this one.