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Digital net properties of a polynomial analogue of Frolov's construction

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A B S T R A C T

Frolov's cubature formula on the unit hypercube has been considered important since it attains an optimal rate of convergence for various function spaces. Its integration nodes are given by shrinking a suitable full rank $\mathbb{Z}$-lattice in $\mathbb{R}^d$ and taking all points inside the unit cube. The main drawback of these nodes is that they are hard to find computationally,

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especially in high dimensions. In such situations, quasi-Monte Carlo (QMC) rules based on digital nets have proven to be successful. However, there is still no construction known that leads to QMC rules which are optimal in the same generality as Frolov's.
In this paper we investigate a polynomial analog of Frolov's cubature formula, which we expect to be important in this respect. This analog is defined in a field of Laurent series with coefficients in a finite field. A similar approach was previously studied in [M. B. Levin. Adelic constructions of low discrepancy sequences. Online Journal of Analytic Combinatorics. Issue 5, 2010.].
We show that our construction is a (t, m, d)-net, which also implies bounds on its star-discrepancy and the error of the corresponding cubature rule. Moreover, we show that our cubature rule is a QMC rule, whereas Frolov's is not, and provide an algorithm to determine its integration nodes explicitly.
To this end we need to extend the notion of (t, m, d)-nets to fit the situation that the points can have infinite digit expansion and develop a duality theory. Additionally, we adapt the notion of admissible lattices to our setting and prove its significance.

1. Introduction

In this paper we consider numerical integration on the d-dimensional unit cube

$$\int_{[0,1]^d} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

approximated by an algorithm using n function evaluations as

$$\sum_{i=1}^{n} w_i \psi(\boldsymbol{x}_i) \quad \text{for } w_i \in \mathbb{R},\ \boldsymbol{x}_i \in [0,1]^d.$$

If the weights satisfy $w_i = n^{-1}$ for all i, the algorithm is called a quasi-Monte Carlo (QMC) rule. For QMC rules, lattice rules (i.e., QMC rules using integration lattices) and digital net rules (i.e., those using digital nets) have been mainly considered, see the books [2,22,27] and the references therein. One intensively studied class of digital net rules is polynomial lattice rules first proposed in [21]. Polynomial lattice rules are a polynomial analog of lattice rules, where polynomial analog means that $\mathbb{R}$ and $\mathbb{Z}$ lattice rules are replaced by a field of Laurent series $\mathbb{F}_b((x^{-1}))$ and a ring of polynomials $\mathbb{F}_b[x]$.

On the other hand, one important non-QMC rule is Frolov's cubature formula, see Section 2. The fascinating property of this cubature rule is that, although the construction is fixed, it attains an optimal rate of convergence for various function spaces. This,