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Kronecker–Halton sequences in $\mathbb{F}_p((X^{-1}))$

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In this paper we investigate the distribution properties of hybrid sequences which are made by combining Halton sequences in the ring of polynomials and digital Kronecker sequences. We give a full criterion for the uniform distribution and prove results on the discrepancy of such hybrid sequences.

1. Preliminaries

Let $(\boldsymbol{z}_n)_{n\geq 0}$ be a sequence in the s-dimensional unit cube $[0,1)^s$, then the *discrepancy* D_N of the first N points of the sequence is defined by

$$D_N = \sup_{B\subseteq[0,1)^s} \left| \frac{A_N(B)}{N} - \lambda(B) \right|$$

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where

$$A_N(B) := \#\{n : 0 \le n < N, \boldsymbol{z}_n \in B\},$$

λ is the s-dimensional Lebesgue measure and the supremum is taken over all axis-parallel subintervals $B \subseteq [0,1)^s$. When restricting the supremum over all axis-parallel subintervals with the lower left point in the origin, then we obtain the *star discrepancy* D_N^* of the first N points of the sequence. It is easy to see that $D_N^* \le D_N \le 2^s D_N^*$. The sequence $(\boldsymbol{z}_n)_{n\ge 0}$ is called *uniformly distributed* if $\lim_{N\to\infty} D_N = 0$.

It is frequently conjectured in the theory of irregularities of distribution, that for every sequence $(\boldsymbol{z}_n)_{n\ge 0}$ in $[0,1)^s$ we have

$$D_N \ge c_s \frac{\log^s N}{N}$$

for a constant $c_s > 0$ and for infinitely many N. In the following we will abbreviate this to $D_N \gg_s \frac{\log^s N}{N}$. Therefore sequences whose discrepancy satisfies $D_N \le C_s \log^s N/N$ for all $N > 1$ with a constant $C_s > 0$ that is independent of N (or $D_N \ll_s \log^s N/N$), are called *low-discrepancy sequences*.

Well-known examples of low-discrepancy sequences are the s-dimensional Halton sequences, digital (t,s)-sequences, and one-dimensional Kronecker sequences $(\{n\alpha\})_{n\ge 0}$ with α irrational and having bounded continued fraction coefficients. For the sake of completeness we define the Halton sequences, the Kronecker sequences, and the digital (t,s)-sequences.

For the Halton sequence [7] $(\boldsymbol{y}_n)_{n\ge 0}$ we choose s different pairwise coprime bases $b_1,\dots,b_s \ge 2$ and construct the ith component $y_n^{(i)}$ of the nth point $\boldsymbol{y}_n = (y_n^{(1)},\dots,y_n^{(s)})$ by representing $n = n_0^{(i)} + n_1^{(i)} b_i + n_2^{(i)} b_i^2 + \cdots$ in base b_i with $0 \le n_j^{(i)} < b_i$ and set

$$y_n^{(i)} = n_0^{(i)}/b_i + n_1^{(i)}/b_i^2 + n_2^{(i)}/b_i^3 + \cdots .$$

The s-dimensional Kronecker sequence related to the real numbers $\alpha_1,\dots,\alpha_s$ is defined by $(\boldsymbol{x}_n = (\{n\alpha_1\}, \dots, \{n\alpha_s\}))_{n\ge 0}$, where $\{\cdot\}$ denotes the fractional part operation. It is well-known to be uniformly distributed if and only if $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Q}$.

For the digital (t,s)-sequences in the sense of Niederreiter [28] we start with the more general, digital $(\boldsymbol{T},s)$-sequences in the sense of Larcher and Niederreiter, see [22].

Definition 1. Choose s, $\mathbb{N}\times\mathbb{N}_0$-matrices $C^{(1)}, \dots, C^{(s)}$ over $\mathbb{F}_p$, p prime. To generate the ith coordinate $x_n^{(i)}$ of $\boldsymbol{x}_n$, represent the integer n in base p

$$n = n_0 + n_1 p + \cdots + n_r p^r \quad \text{with } 0 \le n_j < p,$$

set