Finite Fields and Their Applications 50 (2018) 222–250

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Strongly regular Cayley graphs from partitions of subdifference sets of the Singer difference sets

Koji Momihara [a,1], Qing Xiang [b,*,2]

[a] *Faculty of Education, Kumamoto University, 2-40-1 Kurokami, Kumamoto 860-8555, Japan*
[b] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*

A R T I C L E I N F O

Article history:
Received 15 June 2017
Received in revised form 22 November 2017
Accepted 26 November 2017
Available online xxxx
Communicated by Dieter Jungnickel

MSC:
05E30
05B10

Keywords:
Affine polar graph
i-tight set
m-ovoid
Quadratic form
Singer difference set
Strongly regular graph
Subdifference set

A B S T R A C T

In this paper, we give a new lifting construction of "hyperbolic" type of strongly regular Cayley graphs. Also we give new constructions of strongly regular Cayley graphs over the additive groups of finite fields based on partitions of subdifference sets of the Singer difference sets. Our results unify some recent constructions of strongly regular Cayley graphs related to m-ovoids and i-tight sets in finite geometry. Furthermore, some of the strongly regular Cayley graphs obtained in this paper are new or nonisomorphic to known strongly regular graphs with the same parameters.

* Corresponding author.
E-mail addresses: momihara@educ.kumamoto-u.ac.jp (K. Momihara), qxiang@udel.edu (Q. Xiang).

[1] Research supported by JSPS under Grant-in-Aid for Young Scientists (B) 17K14236 and Scientific Research (B) 15H03636.
[2] Research partially supported by an NSF grant DMS-1600850.

https://doi.org/10.1016/j.ffa.2017.11.010

1. Introduction

We assume that the reader is familiar with the basic theory of strongly regular graphs and difference sets. For strongly regular graphs (srgs), our main references are [3] and [9]. For difference sets, we refer the reader to [10] and Chapter 6 of [2]. Strongly regular graphs are closely related to many other combinatorial/geometric objects, such as two-weight codes, two-intersection sets, m-ovoids, i-tight sets, and partial difference sets. For these connections, we refer the reader to [3, p. 132], [5,12], and some more recent papers [7,6,4] on Cameron-Liebler line classes and hemisystems.

Let Γ be a (simple, undirected) graph. The adjacency matrix of Γ is the $(0,1)$-matrix A with both rows and columns indexed by the vertex set of Γ, where $A_{xy} = 1$ when there is an edge between x and y in Γ and $A_{xy} = 0$ otherwise. A useful way to check whether a graph is strongly regular is by using the eigenvalues of its adjacency matrix. For convenience we call an eigenvalue *restricted* if it has an eigenvector perpendicular to the all-ones vector $\mathbf{1}$. (For a k-regular connected graph, the restricted eigenvalues are the eigenvalues different from k.)

Theorem 1.1. *For a simple graph Γ of order v, not complete or edgeless, with adjacency matrix A, the following are equivalent:*

(i) Γ *is strongly regular with parameters (v, k, λ, μ) for certain integers k, λ, μ,*
(ii) $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$ *for certain real numbers k, λ, μ, where I, J are the identity matrix and the all-ones matrix, respectively,*
(iii) *A has precisely two distinct restricted eigenvalues.*

For a proof of Theorem 1.1, we refer the reader to [3]. An effective method to construct strongly regular graphs is by using Cayley graphs. Let G be an additively written group of order v, and let D be a subset of G such that $0 \notin D$ and $-D = D$, where $-D = \{-d \mid d \in D\}$. The *Cayley graph over G with connection set D*, denoted $\mathrm{Cay}(G, D)$, is the graph with the elements of G as vertices; two vertices are adjacent if and only if their difference belongs to D. In the case when $\mathrm{Cay}(G, D)$ is a strongly regular graph, the connection set D is called a (regular) *partial difference set*. Examples of strongly regular Cayley graphs are the Paley graphs $\mathrm{P}(q)$, where q is a prime power congruent to 1 modulo 4, the Clebsch graph, and the affine orthogonal graphs ([3]). For $\Gamma = \mathrm{Cay}(G, D)$ with G abelian, the eigenvalues of Γ are exactly $\chi(D) := \sum_{d \in D} \chi(d)$, where χ runs through the character group of G. This fact reduces the problem of computing eigenvalues of abelian Cayley graphs to that of computing some character sums, and is the underlying reason why the Cayley graph construction has been very effective for the purpose of constructing srgs. The survey of Ma [12] contains much of what is known about partial difference sets and about connections with strongly regular graphs.

A (v, k, λ, μ) srg is said to be of *Latin square type* (respectively, *negative Latin square type*) if $(v, k, \lambda, \mu) = (n^2, r(n - \epsilon), \epsilon n + r^2 - 3\epsilon r, r^2 - \epsilon r)$ and $\epsilon = 1$ (respectively, $\epsilon = -1$).