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On the next-to-minimal weight of projective Reed–Muller codes

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ABSTRACT

In this paper we present several values for the next-to-minimal weights of projective Reed–Muller codes. We work over $\mathbb{F}_q$ with $q \geq 3$ since in [3] we have determined the complete values for the next-to-minimal weights of binary projective Reed–Muller codes. As in [3] here we also find examples of codewords with next-to-minimal weight whose set of zeros is not in a hyperplane arrangement.

1. Introduction

Reed–Muller codes were introduced in 1954 by D.E. Muller ([13]) as codes defined over $\mathbb{F}_2$, and a decoding algorithm for them was devised by I.S. Reed ([14]). In

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1968 Kasami, Lin, and Peterson ([8]) extended the original definition to a finite field $\mathbb{F}_q$, where q is any prime power, and named these codes "generalized Reed–Muller codes". They also presented some results on the weight distribution, the dimension of the codes being determined in later works. In coding theory one is always interested in the values of the higher Hamming weights of a code because of their relationship with the code performance, but usually this is not a simple problem. For the generalized Reed–Muller codes, the complete determination of the second lowest Hamming weight, also called next-to-minimal weight, was only completed in 2010, when Bruen ([2]) observed that the value of these weights could be obtained from unpublished results in the Ph.D. thesis of D. Erickson ([4]) and Bruen's own results from 1992 and 2006. Now that Bruen called the attention to Erickson's thesis we know that the complete list of the next-to-minimal weights for the generalized Reed–Muller codes may be obtained by combining results from Erickson's thesis with results by Geil (see [6]) or with results by Rolland (see [18]).

In 1990 Lachaud introduced the class of projective Reed–Muller codes (see [9]). The parameters of these codes were determined by Serre ([21]), for some cases, and by Sørensen ([22]) for the general case. As for the determination of the next-to-minimal weight for these codes, there are some results (also about higher Hamming weights) on this subject by Rodier and Sboui ([16], [17], [20]) and also by Ballet and Rolland ([1]). Recently the next-to-minimal weights of projective Reed–Muller codes defined over $\mathbb{F}_2$ were completely determined (see [3]). In this paper we present several results on the next-to-minimal Hamming weights for projective Reed–Muller codes defined over $\mathbb{F}_q$, for $q \geq 3$, including the complete determination of the next-to-minimal weights for the case of projective Reed–Muller codes defined over $\mathbb{F}_3$. In the next section we recall the definitions of the generalized and projective Reed–Muller codes, and some results of geometrical nature that will allow us to determine many cases of higher Hamming weights for the projective Reed–Muller codes, which we do in the last section.

2. Preliminary results

Let $\mathbb{F}_q$ be a finite field and let $I_q = (X_1^q - X_1, \ldots, X_n^q - X_n) \subset \mathbb{F}_q[X_1, \ldots, X_n]$ be the ideal of polynomials which vanish at all points $P_1, \ldots, P_{q^n}$ of the affine space $\mathbb{A}^n(\mathbb{F}_q)$. Let $\varphi : \mathbb{F}_q[X_1, \ldots, X_n]/I_q \to \mathbb{F}_q^{q^n}$ be the $\mathbb{F}_q$-linear transformation given by $\varphi(g + I_q) = (g(P_1), \ldots, g(P_{q^n}))$.

Definition 2.1. Let d be a nonnegative integer. The generalized Reed–Muller code of order d is defined as $\mathrm{RM}(n, d) = \{\varphi(g + I_q) \mid g = 0 \text{ or } \deg(g) \leq d\}$.

It is not difficult to prove that $\mathrm{RM}(n, d) = \mathbb{F}_q^{q^n}$ if $d \geq n(q-1)$, so in this case the minimum distance is 1. Let $d \leq n(q-1)$ and write $d = a(q-1) + b$ with $0 < b \leq q-1$, then the minimum distance of $\mathrm{RM}(n, d)$ is

$$W^{(1)}_{\mathrm{RM}}(n, d) = (q - b)q^{n-a-1}.$$