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Asymptotic vanishing of homogeneous components of multigraded modules and its applications

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In this article, we give a condition on the vanishing of finitely many homogeneous components which must imply the asymptotic vanishing for multigraded modules. We apply our result to multi-Rees algebras of ideals. As a consequence, we obtain a result on normality of monomial ideals, which extends and improves the results of Reid, Roberts and Vitulli [14], Singla [16], and Sarkar and Verma [15].

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and let R be a Noetherian standard $\mathbb{N}^r$-graded ring with $R_{\mathbf{0}} = A$, i.e., R is generated in degrees $\boldsymbol{e}_1, \dots, \boldsymbol{e}_r$ over A with $R_{\boldsymbol{e}_i} \neq (0)$ for all $i = 1, \dots, r$. Let M be a finitely generated $\mathbb{Z}^r$-graded R-module and let N be a graded R-submodule of M. In this article, we study the asymptotic vanishing property of homogeneous components $[M/N]_{\boldsymbol{n}}$ of the quotient M/N. Particularly, we give a condition on the vanishing.

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Our main motivation comes from the following facts on the normality of monomial ideals. Let $S = k[X_1, \dots, X_d]$ be a polynomial ring over a field k and let I be a monomial ideal in S. In 2003, Reid, Roberts and Vitulli [14] proved that if $I, I^2, \dots, I^{d-1}$ are integrally closed, then all the powers of I are integrally closed, i.e., I is a normal ideal. In 2007, Singla improved this result. Let $\ell = \lambda(I)$ be the analytic spread of I. Then Singla [16] proved that I is a normal ideal if $I, I^2, \dots, I^{\ell-1}$ are integrally closed. Note that the inequality $\ell \le d$ always holds true. On the other hand, more recently, Sarkar and Verma [15] gave the other generalization of the original result of Reid, Roberts and Vitulli [14]. Let $I_1, \dots, I_r$ be $(X_1, \dots, X_d)$-primary monomial ideals in S. They proved that if $\boldsymbol{I}^{\boldsymbol{n}} = I_1^{n_1} \cdots I_r^{n_r}$ is integrally closed for any $\boldsymbol{n} \in \mathbb{N}^r$ with $1 \le |\boldsymbol{n}| \le d-1$, where $|\boldsymbol{n}| = n_1 + \cdots + n_r$ denotes the sum of all entries of a vector $\boldsymbol{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, then all the power products $\boldsymbol{I}^{\boldsymbol{n}}$ are integrally closed. Thus, the original result of Reid, Roberts and Vitulli [14] was partially generalized to finitely many monomial ideals. The approach in [15] is different from the ones in [14,16]. In [15], the authors used the local cohomology modules of the multi-Rees algebra $\mathcal{R}(\boldsymbol{I})$ of ideals $I_1, \dots, I_r$ and its normalization $\overline{\mathcal{R}}(\boldsymbol{I})$ instead of convex geometry as in [14,16].

The purpose of this article is to extend and improve all of the above results with a more general approach inspired by [15]. In order to state our results, let us recall some notations associated to multigraded modules.

For a Noetherian standard $\mathbb{N}^r$-graded ring R with $R_{\boldsymbol{0}} = A$, let $R_{++} = R_{\boldsymbol{e}}R$ be the irrelevant ideal of R where $\boldsymbol{e} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_r = (1, \dots, 1) \in \mathbb{N}^r$. Let $R_+ = (R_{\boldsymbol{e}_1}, \dots, R_{\boldsymbol{e}_r})R$. Note that $R_{++} = R_+$ when $r = 1$. Let $\mathfrak{M} = \mathfrak{m}R + R_+$ be the homogeneous maximal ideal of R. For a finitely generated R-module M, we set

$$\operatorname{Supp}_{++}(M) = \{P \in \operatorname{Spec} R \mid M_P \neq (0), P \text{ is graded and } R_{++} \nsubseteq P\}.$$

Then the spread $s(M)$ of M introduced by Kirby and Rees [10] is defined to be

$$s(M) = \dim \operatorname{Supp}_{++}(M/\mathfrak{m}M) + 1.$$

Here we set $\dim \emptyset = -1$. Note that for an ideal I in A, the spread $s(\mathcal{R}(I))$ of the Rees algebra of I is just the analytic spread $\lambda(I)$ of the ideal I. For any $i = 1, \dots, r$, let

$$a^i(M) = \sup\{k \in \mathbb{Z} \mid [H_{\mathfrak{M}}^{\dim M}(M)]_{\boldsymbol{n}} \neq (0) \text{ for some } \boldsymbol{n} \in \mathbb{Z}^r \text{ with } n_i = k\},$$

where $H_{\mathfrak{M}}^i(M)$ is the ith local cohomology module of M with respect to $\mathfrak{M}$ in the category of $\mathbb{Z}^r$-graded R-modules. Then the $\boldsymbol{a}$-invariant vector is defined to be

$$\boldsymbol{a}(M) = (a^1(M), \dots, a^r(M)) \in \mathbb{Z}^r.$$

Then our main result can be stated as follows.