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The Zariski topology on sets of semistar operations without finite-type assumptions

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ABSTRACT

We study, from a topological point of view, spaces of semistar operations without the hypothesis that they are of finite type. In particular, we analyze when the space of all semistar operations, the space of spectral operations, and the set of stable operations are spectral spaces.

1. Introduction

Semistar operations were introduced by Okabe and Matsuda in [18] as a generalization of the previous concept of star operations, whose origin traces back to Krull [16]. In [12] and [10], the set SStar(D) of semistar operations on an integral domain D was endowed with a Zariski-like topology, and thus considered from a topological point of view. In particular, it was shown that some distinguished subsets of SStar(D) are *spectral spaces*

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[13], that is, they are homeomorphic to the prime spectrum of a commutative ring: more precisely, this was shown for the set of finite-type operations [12, Theorem 2.13], of finite-type spectral operations [10, Theorem 4.6] and of finite-type eab operations [10, Theorem 5.11(3)] (the definitions will be recalled in Section 2). These proofs were carried out by using an ultrafilter-theoretic criterion for spectral spaces (introduced in [8]) and the possibility of writing the supremum of a family of finite-type operations in a relatively explicit way (proved in [2, p. 1628] in the setting of star operations).

In this paper, we continue this study outside the finite-type case, showing that several natural-looking spaces need not to be spectral, and trying to characterize when they are spectral. More in detail, we study the set of all semistar operations in Section 4, the set of stable operations in Section 5, and the set of spectral operations in Section 6. While we use an ultrafilter-based proof in Theorem 3.2 (and subsequently use this result in Theorems 4.10 and 5.5), we shall often use a more bare-handed approach, based on the compactness of the constructible topology [13, Proposition 4] (on which also the ultrafilter criterion is based), which is tested through the finite intersection property of the closed sets.

2. Preliminaries

2.1. Semistar operations

In the paper, we shall always use D to denote an integral domain, and K to denote its quotient field. We will denote by $\mathbf{F}(D)$ the set of D-submodules of K, and by $\mathcal{F}(D)$ the set of fractional ideals of D. For basic properties of semistar operations the reader may consult [18].

A *semistar operation* on D is a map $\star : \mathbf{F}(D) \longrightarrow \mathbf{F}(D)$, $I \mapsto I^\star$, such that, for every $I, J \in \mathbf{F}(D)$ and every $z \in K$, the following properties hold:

- $I \subseteq I^\star$ ($\star$ is *extensive*);
- $(I^\star)^\star = I^\star$ ($\star$ is *idempotent*);
- if $I \subseteq J$, then $I^\star \subseteq J^\star$ ($\star$ is *order-preserving*);
- $z \cdot I^\star = (zI)^\star$.

The set of all semistar operations on D is denoted by $\mathrm{SStar}(D)$; it is a complete lattice, where $\inf \Delta$ is the semistar operation such that

$$I^{\inf \Delta} = \bigcap_{\star \in \Delta} I^\star \quad \text{for every } I \in \mathbf{F}(D),$$

and $\sup \Delta$ is the semistar operation $\sharp$ such that $I = I^\sharp$ if and only if $I = I^\star$ for every $\star \in \Delta$.

Given a semistar operation $\star$, its *quasi-spectrum* (denoted by $\mathrm{QSpec}^\star(D)$) is the set of prime ideals P of D such that $P = P^\star \cap D$.