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The Koszul property for graded twisted tensor products

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ABSTRACT

Let $\mathbb{K}$ be a field. Let A and B be connected $\mathbb{N}$-graded $\mathbb{K}$-algebras. Let C denote a twisted tensor product of A and B in the category of connected $\mathbb{N}$-graded $\mathbb{K}$-algebras. The purpose of this paper is to understand when C possesses the Koszul property, and related questions. We prove that if A and B are quadratic, then C is quadratic if and only if the associated graded twisting map has a property we call the unique extension property. We show that A and B being Koszul does not imply C is Koszul (or even quadratic), and we establish sufficient conditions under which C is Koszul whenever both A and B are. We analyze the unique extension property and the Koszul property in detail in the case where $A = \mathbb{K}[x]$ and $B = \mathbb{K}[y]$.

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1. Introduction

Though the study of factorization structures in mathematics has a long history, there has been much recent interest in very general questions about the dual notions of product and factorization for associative algebras over a field. Čap, Schichl, and Vanžura introduced in [13] a very general notion of product for a pair of associative algebras, or equivalently, a notion of factorization, called the twisted tensor product. This product is analogous to the Zappa–Szép product for groups, see [4] for example, and the bicrossed product for Hopf algebras, [1]. Commutative tensor products, Ore extensions, and smash products of algebras can all be realized as particular cases of twisted tensor products. Homological and ring-theoretic properties of these particular cases are well-studied, and a number of recent papers establish such properties of twisted tensor products in the graded setting; see [9], [12], [14] for example.

One particularly important homological property of a graded algebra is the Koszul property, introduced by Priddy in [11]. Let $\mathbb{K}$ be a field. Let A be an $\mathbb{N}$-graded $\mathbb{K}$-algebra. We further assume A is connected ($\dim_{\mathbb{K}} A_0 = 1$) and locally finite dimensional ($\dim_{\mathbb{K}} A_i < \infty$ for all $i \geq 0$). For a $\mathbb{K}$-vector space V, let $T(V)$ denote the tensor algebra generated by V.

The graded algebra A is called *one-generated* if the canonical multiplication map $\pi : T(A_1) \to A$ is surjective. Let $I = \langle (\ker \pi) \cap A_1 \otimes A_1 \rangle$ be the ideal of $T(A_1)$ generated by the degree 2 elements of the kernel of π. If A is one-generated, the *quadratic part of* A is the algebra $q(A) = T(A_1)/I$. If $A \cong q(A)$ then A is called *quadratic*.

The graded algebra A is called *Koszul* if the trivial module $\mathbb{K} = A_0 = A/A_+$ admits a resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{K} \to 0$$

such that each P_i is a graded free left A-module generated in degree i.

Two well-known facts follow immediately from this definition:

(1) every Koszul algebra is quadratic,
(2) every one-generated, free algebra is Koszul.

Many important quadratic algebras arising naturally in mathematics are Koszul, and Koszul algebras satisfy a powerful duality property that sometimes underlies deep connections between apparently unrelated problems. One famous example is the duality between the symmetric algebra and the exterior algebra underlying the so-called *BGG correspondence* [2]. For additional motivation, background, and examples of Koszul algebras, we encourage the interested reader to consult the book [10], which remains the most comprehensive and useful treatment of the theory of quadratic and Koszul algebras we have read.