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Higher commutator theory for congruence modular varieties ☆

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ABSTRACT

We develop the basic properties of the higher commutator for congruence modular varieties.

1. Introduction

This article develops some basic properties of a congruence lattice operation, called the higher commutator, for varieties of algebras that are congruence modular. The higher commutator is a higher arity generalization of the binary commutator, which was first defined in full generality in the seventies. While the binary commutator has a rich theory for congruence modular varieties, the theory of the higher arity commutator was poorly understood outside of the context of congruence permutability.

We begin by discussing the evolution of centrality in Universal Algebra. Centrality is easily understood in groups as the commutativity of multiplication. Here, it plays an essential role in defining important group-theoretic notions such as abelianness, solvabil-

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ity, nilpotence, etc. Naturally, a systematic calculus to study centrality was developed. For a group G and $a, b \in G$, the group commutator of a and b is defined to be

$$[a, b] = a^{-1}b^{-1}ab.$$

Actually, one can go further and use group commutators to define a very useful operation on the lattice of normal subgroups of G.

Definition 1.1. Suppose that G is a group and M and N are normal subgroups of G. The group commutator of M and N is defined to be

$$[M, N] = \mathrm{Sg}_G(\{[m, n] : m \in M, n \in N\}),$$

where $\mathrm{Sg}_G(S)$ denotes the subgroup generated by the elements belonging to S. It is easily checked that $[M, N]$ is a normal subgroup of G.

Suppose that $f : G \to H$ is a surjective homomorphism and $\{N_i : i \in I\}$ are normal subgroups of G. The following properties are easy consequences of Definition 1.1, where $\wedge$ and $\vee$ denote the operations of meet and join in the lattice of normal subgroups of G:

1. $[M, N] \subseteq M \wedge N$,
2. $[f(M), f(N)] = f([M, N])$,
3. $[M, N] = [N, M]$,
4. $[M, \bigvee_{i \in I} N_i] = \bigvee_{i \in I} [M, N_i]$,
5. For any normal subgroup K of G contained in $M \wedge N$, the elements of M/K commute with N/K if and only if $[M, N] \subseteq K$.

Rings have an analogous commutator theory. For two ideals I, J of a ring R the commutator is $[I, J] = IJ - JI$. This operation satisfies the same basic properties as the commutator for groups and allows one to analogously define abelian, solvable and nilpotent rings. As it turns out, the notion of centrality and the existence of a well-behaved commutator operation is not an idiosyncrasy of groups or rings. In [13], J.D.H. Smith defined a language-independent type of centrality that generalized the known examples. He then used this definition to show that any algebra belonging to a Mal'cev variety came equipped with a commutator as powerful as the commutator for groups or rings.

Joachim Hagemann and Christian Herrmann later extended the results of Smith to congruence modular varieties in [8]. The language-independent definition of centrality allows for language-independent definitions of abelianness and related notions such solvability and nilpotence. The existence of a robust commutator for a congruence modular variety means that these definitions are powerful and well-behaved, and provide an important tool to study the consequences of congruence modularity. For example, quotients of abelian algebras that belong to a modular variety are abelian, but this need not be true in general.