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Incidence graphs and non-negative integral quadratic forms

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We present directed multigraphs (quivers) and their walks as combinatorial models for non-negative semi-unit integral quadratic forms of Dynkin type $\mathbb{A}$ and their associated roots and radical vectors. This point of view leads to an easy description of those forms which are positive and weakly positive (properties of interest in Lie theory and the representation theory of algebras), and derives into a categorification of such unitary forms.

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1. Introduction

Unitary integral quadratic forms, degree two polynomials

$$q(x_1,\dots,x_n) = x_1^2 + \cdots + x_n^2 + \sum_{1\leq i<j\leq n} q_{ij}x_ix_j,$$

with integral coefficients q_{ij}, have been a guiding light in mathematics, in particular in algebra, since the mid-20th century. They appear in Lie theory as trace and Killing forms [12,3], in representation theory of associative algebras as Tits and homological (Euler) forms [11,7], and in many other subjects as singularities, the theory of representations of posets and bocses, and group theory, to mention a few examples. Properties of quadratic forms (and their root systems) control or reflect properties of the algebraic structure from where they come. In turn, these fields have inspired a diversity of aspects and methods around integral quadratic forms such as weak positivity and non-negativity, flations, reflections, one-point extensions and edge reductions, as well as many interesting computational algorithms (see for instance [3–5,7,14,15,9]).

Classification of non-negative semi-unitary integral quadratic forms up to $\mathbb{Z}$-equivalence is known. It is given in [5] in terms of Dynkin type and corank, and we recall it in Theorem 3.2. Recently an alternative classification for this class of quadratic forms was obtained in [13] in terms of so-called canonical r-vertex extensions of Dynkin graphs.

It is sometimes convenient to switch to a combinatorial and graphical setting for both unitary quadratic forms and their root systems (as done for instance in [1] and [2]). This perspective extends into a categorical framework for non-negative unitary forms, which surprisingly had not been considered before, as far as the author is aware. The case of quadratic forms with Dynkin type $\mathbb{A}$ is discussed in this paper.

We use as combinatorial model for semi-unitary quadratic forms (those homogeneous polynomials of degree two with diagonal coefficients in the set $\{0,1\}$, in contrast to only 1 for the unitary case) a *directed multigraph* or *quiver* Q. For a walk in Q with shape

$$\bullet_i \xrightarrow{a} \bullet \xleftarrow{b} \bullet \xrightarrow{c} \bullet_j$$

we use the notation $ab^{-1}c$ for the walk from i to j, and $c^{-1}ba^{-1}$ for the reversed walk (see precise definitions in 2.2 below). Consider variables $x = (x_c)_{c\in Q_1}$ indexed by the set of arrows Q_1 of Q, and the following integral quadratic form

$$q_Q(x) = \sum_{c,d\in Q_1} \langle c,d\rangle x_cx_d, \tag{1}$$