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ON QUILLEN'S CONJECTURE FOR p-SOLVABLE GROUPS

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Abstract. We give a new proof of Quillen's conjecture for solvable groups via a geometric and explicit method. For p-solvable groups, we provide both a new proof using the Classification of Finite Simple Groups and an asymptotic version without employing it.

1. Introduction

Let G be a finite group, let p be a prime and let $\mathcal{A}_p(G)$ be the poset consisting of the non-trivial elementary abelian p-groups of G ordered by inclusion. The homotopy properties of the topological realization $|\mathcal{A}_p(G)|$ were first studied in [9]. There, Quillen introduced the following conjecture, where we denote by $O_p(G)$ the largest normal p-subgroup of G:

Conjecture 1.1 (Quillen's conjecture). *If $|\mathcal{A}_p(G)|$ is contractible then $O_p(G) \neq 1$.*

If the implication 1.1 holds for G we say that G *satisfies* $\mathcal{QC}$. In [2, p.474], Ashchbacher and Smith introduced the following notion, which we denote by $\mathcal{QD}_p$.

Definition 1.2. Let G be a finite group of p-rank r. We say that G has *Quillen dimension at p* if $O_p(G) = 1 \Rightarrow \widetilde{H}_{r-1}(|\mathcal{A}_p(G)|;\mathbb{Q}) \neq 0$.

Note that $r-1$ is the top dimension $*$ for which $\widetilde{H}_*(|\mathcal{A}_p(G)|;\mathbb{Q})$ can possibly be non-zero. As contractibility leads to zero homology it is clear that

$$\mathcal{QD}_p \text{ holds for } G \Rightarrow \mathcal{QC} \text{ holds for } G.$$

In fact, Quillen proved that $\mathcal{QD}_p$ holds for G solvable [9, Corollary 12.2] and Alperin proved that $\mathcal{QD}_p$ holds for G p-solvable via the Classification of the Finite Simple Groups (CFSG), see [3, Theorem 1], [10, Theorem 8.2.12] and [2, Theorem 0.5].

In this work, we provide new proofs for the solvable and the p-solvable cases of Quillen's conjecture. We also give an asymptotic version for the p-solvable case that does not use the CFSG. Our arguments are of geometric nature and the non-zero top dimensional homology class that we construct belongs to certain subgroup $\widetilde{H}^c_{r-1}(|\mathcal{A}_p(G)|;\mathbb{Q})$ of $\widetilde{H}_{r-1}(|\mathcal{A}_p(G)|;\mathbb{Q})$. We term the elements of this subgroup *constructible classes* and they involve only the top two layers of the poset $\mathcal{A}_p(G)$, i.e., Sylow subgroups and their hyperplanes. Accordingly, we define the following condition, which we denote by $\mathcal{QD}^c_p$:

Definition 1.3. Let G be a finite group of p-rank r. We say that G has *constructible Quillen dimension at p* if $O_p(G) = 1 \Rightarrow \widetilde{H}^c_{r-1}(|\mathcal{A}_p(G)|;\mathbb{Q}) \neq 0$.

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