Journal of Algebra 513 (2018) 368–387

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On low Gorenstein colength

J. Elias [1,*], R. Homs [2]

Departament d'Àlgebra i Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

ARTICLE INFO

Article history:
Received 6 February 2017
Available online 31 July 2018
Communicated by Luchezar L. Avramov

MSC:
primary 13H10
secondary 13H15, 14C05

Keywords:
Gorenstein rings
Artin rings
Macaulay's inverse system
Hilbert function
Gorenstein colength

ABSTRACT

The aim of this paper is to extend the characterization of Teter rings of [9] to Artin rings of Gorenstein colength two and to improve and complete the result [1, Theorem 5.5] on such rings by using Macaulay's inverse system. We provide several explicit examples and families of Gorenstein colength two rings taking into account their analytic type.

1. Introduction

In this paper we deal with Artinian local **k**-algebras where **k** is an arbitrary field. Hence, if A is such an algebra then we may assume that A is a quotient of the ring of formal power series $R = \mathbf{k}[\![x_1, \ldots x_n]\!]$ for some integer n. We assume that $n \geq 1$ and we

* Corresponding author.
E-mail addresses: elias@ub.edu (J. Elias), rhomspon7@ub.edu (R. Homs).

[1] Partially supported by MTM2013-40775-P.
[2] Partially supported by MTM2013-40775-P and BES-2014-069364.

https://doi.org/10.1016/j.jalgebra.2018.07.016

denote by $\mathfrak{m} = (x_1, \ldots, x_n)$ the maximal ideal of R and by $\mathfrak{n} = \mathfrak{m}/I$ the maximal ideal of $A = R/I$.

Recall that the length of the canonical module ω_A agrees with the length of A. Hence the ring is Gorenstein if and only if there is an isomorphism of A-modules $\varphi : \omega_A \longrightarrow A$ or, equivalently, an epimorphism $\varphi : \omega_A \longrightarrow A$. Teter characterized Artin local rings $A = R/I$ of the form $G/\operatorname{soc}(G)$, where G is a Gorenstein ring and $\operatorname{soc}(G)$ is its socle, [17]. From now on we will call such rings Teter rings, [16]. Huneke and Vraciu improved this result by proving that A is Teter if and only if there is an epimorphism $\varphi : \omega_A \longrightarrow \mathfrak{n}$, under some suitable hypothesis; see also [3]. In [9] we improved these results by using Macaulay's inverse system device, see Theorem 4.4.

The Gorenstein colength $\operatorname{gcl}(A)$ of an Artin ring A is defined as the minimum $\ell(G) - \ell(A)$ where G is an Artin Gorenstein ring such that $A \simeq G/H$ for some ideal $H \subset G$, see Definition 3.3. Notice that A is Gorenstein if and only if $\operatorname{gcl}(A) = 0$. Moreover, if A is not Gorenstein, then it is Teter if and only if $\operatorname{gcl}(A) = 1$.

In the main result of this paper, Theorem 4.6, we extend and improve the characterization of Artin rings $A = R/I$ of Gorenstein colength two of [1, Theorem 5.5] in terms of its Macaulay's inverse system $I^\perp$. In particular, we prove that $I^\perp$ is generated by $x_1 \circ F, \ldots, x_{n-1} \circ F, x_n^2 \circ F$ for a suitable polynomial F up to an analytic isomorphism of R. One of the main problems in this area is the computation of the Gorenstein colength, see [2]. Theorem 4.6 enables us to construct explicit examples and families of Gorenstein colength two rings.

In section two we recall some basic results on Macaulay's inverse systems. Section three is devoted to the study of Artin rings of low Gorenstein colength, i.e. of Gorenstein colength at most two; the main result of this section is Proposition 3.9. We end the section by providing examples that answer several natural questions on Gorenstein covers. In section four we prove the main result of this paper where we characterize Artin rings of Gorenstein colength two and we describe its Macaulay's inverse systems, Theorem 4.6. We finish the paper by giving several explicit examples and families of Gorenstein colength two rings.

We perform the computations of this paper by using the Singular library [6], [5]. See [1], [2] and [9], for more results on Teter rings, Gorenstein colength and related problems.

2. Preliminaries

Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n} = \mathfrak{m}/I$. We denote by $E_A(\mathbf{k})$ the injective hull of the residue field $\mathbf{k}$.

The Hilbert function of A is the numerical function $\operatorname{HF}_A : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $\operatorname{HF}_A(i) = \dim_{\mathbf{k}}(\mathfrak{n}^i/\mathfrak{n}^{i+1})$, $i \geq 0$. The embedding dimension of A is $\operatorname{embd}(A) = \operatorname{HF}_A(1)$. The socle degree of A, denoted by $\operatorname{socdeg}(A)$, is the last integer s such that $\operatorname{HF}_A(s) \neq 0$. The socle of A is the $\mathbf{k}$-vector subspace $\operatorname{soc}(A) = (0 :_A \mathfrak{n})$ of A, and the Cohen–Macaulay type of A is $\tau(A) = \dim_{\mathbf{k}}(\operatorname{soc}(A))$. Recall that A is Gorenstein if and only if $\tau(A) = 1$. We denote by $\ell(M)$ the length of an A-module M.