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Deformations of algebras defined by tilting bundles

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In this paper we produce noncommutative algebras derived equivalent to deformations of schemes with tilting bundles. We do this in two settings, first proving that a tilting bundle on a scheme lifts to a tilting bundle on an infinitesimal deformation of that scheme and then expanding this result to $\mathbb{C}^*$-equivariant deformations over schemes with a good $\mathbb{C}^*$-action. In both these situations the endomorphism algebra of the lifted tilting bundle produces a deformation of the original endomorphism algebra, and this is a graded deformation in the $\mathbb{C}^*$-equivariant case.

We apply our results to rational surface singularities, generalising the deformed preprojective algebras of Crawley-Boevey and Holland, and also to symplectic situations where the deformations produced are related to symplectic reflection algebras.

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1. Introduction

1.1. Overview

A tilting bundle T on a scheme X induces an equivalence between the derived category of quasicoherent sheaves on X and the derived category of right modules for the algebra $A := \mathrm{End}_X(T)$. This is a valuable tool which allows homological properties to be understood and calculated in either the algebraic or geometric setting. Deformation theory is controlled by homological data, and if a scheme is derived equivalent to an algebra then its deformations should be derived equivalent to certain deformations of the algebra. It is anticipated that a tilting bundle should lift to deformations of the scheme and that the endomorphism algebra of the lifted tilting bundle should produce deformations of the algebra [10, Section 3.8]. We will prove this result both for deformations over complete local Noetherian schemes and for $\mathbb{C}^*$-equivariant deformations over affine schemes with unique $\mathbb{C}^*$-invariant closed point.

These results can be viewed from two intertwined perspectives. On the one hand they produce noncommutative algebras derived equivalent to geometric deformations, and so construct noncommutative counterparts to interesting geometric phenomenon. One example is the *simultaneous resolution of the Artin component* for rational surface singularities. In this situation a rational surface singularity has a versal deformation, and the Artin component of this has a simultaneous resolution after base change which can be realised as the versal deformation of the minimal resolution [1,9,31,51]. Tilting bundles on minimal resolutions of rational surface singularities were constructed in [53] and the existence of a noncommutative algebra derived equivalent to the simultaneous resolution was proposed by Riemenschneider [40]. We discuss the construction of such algebras as one of our applications below. Further, realising these algebras as deformations makes it possible to actually calculate them in many examples.

On the other hand, the results construct deformations of noncommutative algebras via geometric techniques, with the advantage of finding exactly the deformations of noncommutative algebras that correspond to geometric deformations. For G a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ the skew group algebras $\mathbb{C}[x_1, \dots, x_n] \rtimes G$ always provide noncommutative crepant resolutions of the quotient singularities $\mathbb{C}^n/G$ [48], and as Koszul algebras their PBW deformations are classified by the results of Braverman and Gaitsgory [6]. However, considering the case of a small, finite subgroup of $\mathrm{GL}_2(\mathbb{C})$ then minimal resolutions exist but if G is not a subgroup of $\mathrm{SL}_2(\mathbb{C})$ then the skew group algebras are not derived equivalent to the minimal resolutions and have no PBW deformations [28, Theorem 5.0.6.]