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Lefschetz invariants and Young characters for representations of the hyperoctahedral groups

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The ring $R(B_n)$ of virtual $\mathbb{C}$-characters of the hyperoctahedral group B_n has two $\mathbb{Z}$-bases consisting of permutation characters, and the ring structure associated with each basis of them defines a partial Burnside ring of which $R(B_n)$ is a homomorphic image. In particular, the concept of Young characters of B_n arises from a certain set $\mathcal{U}_n$ of subgroups of B_n, and the $\mathbb{Z}$-basis of $R(B_n)$ consisting of Young characters, which is presented by L. Geissinger and D. Kinch [7], forces $R(B_n)$ to be isomorphic to a partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$. The linear $\mathbb{C}$-characters of B_n are analyzed with reduced Lefschetz invariants which characterize the unit group of $\Omega(B_n, \mathcal{U}_n)$. The parabolic Burnside ring $\mathcal{PB}(B_n)$ is a subring of $\Omega(B_n, \mathcal{U}_n)$, and the unit group of $\mathcal{PB}(B_n)$ is isomorphic to the four group. The unit group of the parabolic Burnside ring of the even-signed permutation group D_n is also isomorphic to the four group.

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1. Introduction

Let G be a finite group, and let G-**set** be the category of finite left G-sets and G-equivariant maps. The Burnside ring $\Omega(G)$, which is the Grothendieck ring of the category G-**set**, is the commutative unital ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes $[X]$ of finite left G-sets X with disjoint union for addition and cartesian product for multiplication. We denote by $R(G)$ the ring of virtual $\mathbb{C}$-characters of G. Set $[n] = \{1, 2, \ldots, n\}$, and let S_n be the symmetric group on $[n]$. We denote by $\mathcal{Y}_n$ the set of Young subgroups of S_n, which is closed under intersection and conjugation. By [15, §7], $\Omega(S_n)$ possesses the partial Burnside ring $\Omega(S_n, \mathcal{Y}_n)$ relative to the Young subgroups as a subring, and $\Omega(S_n, \mathcal{Y}_n) \cong R(S_n)$. This fact means that the characters $1_Y^{S_n}$ induced from the trivial characters 1_Y of Y for $Y \in \mathcal{Y}_n$ form a $\mathbb{Z}$-basis of $R(S_n)$ (see, *e.g.*, [2, Proposition 3]). Let C_2 be a cyclic group of order 2, and let V_n be the direct product $C_2^{(n)}$ of n copies of C_2. We denote by B_n the hyperoctahedral group, that is, the wreath product $C_2 \wr S_n$ defined to be a semidirect product $V_n \rtimes S_n$ of V_n with S_n. Let $\mathcal{Z}_n$ be the set of all products KY of $K \leq V_n$ and $Y \in \mathcal{Y}_n$ with $|V_n : K| \leq 2$ and $Y \leq N_{S_n}(K)$. We establish in §3 that $R(B_n)$ is a homomorphic image of the partial Burnside ring $\Omega(B_n, \widetilde{\mathcal{Z}}_n)$ relative to the set $\widetilde{\mathcal{Z}}_n$ of intersections of subgroups contained in $\mathcal{Z}_n$.

For a ring R, we denote by $R^\times$ the unit group of R. By [13, Example 2], $R(S_n)^\times$ is isomorphic to the four group. There exists a unit of $\Omega(S_n, \mathcal{Y}_n)$ which enables us to describe the sign character $\mathrm{sgn}_n : S_n \to \mathbb{C}$ as a $\mathbb{Z}$-linear combination of the characters $1_Y^{S_n}$ for $Y \in \mathcal{Y}_n$ (see [2, Corollary 2] and [9, §4]); such a description is called Solomon's formula. The ring $R(B_n)$ includes exactly four linear $\mathbb{C}$-characters, and $R(B_n)^\times$ is generated by the nontrivial linear $\mathbb{C}$-characters and -1_{B_n}. In §4 we identify $R(B_n)^\times$ with a subgroup of $\Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times$, and then describe the linear $\mathbb{C}$-characters of B_n as $\mathbb{Z}$-linear combinations of the characters $1_H^{B_n}$ for $H \in \mathcal{Z}_n$.

There is a set $\mathcal{U}_n$ of subgroups of B_n such that the characters $1_H^{B_n}$ for $H \in \mathcal{U}_n$ form a $\mathbb{Z}$-basis of $R(B_n)$ (cf. [7, Corollary II.4]). In §5 we define the partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$ relative to the Young subgroups of B_n, which is a subring of $\Omega(B_n)$ isomorphic to $R(B_n)$. The parabolic Burnside ring $\mathcal{PB}(B_n)$ (cf. [1, §4]) is a subring of $\Omega(B_n, \mathcal{U}_n)$. By [4, (66.29) Corollary], the sign character $\varepsilon_n : B_n \to \mathbb{C}$ is described as a $\mathbb{Z}$-linear combination of the characters $1_H^{B_n}$ for parabolic subgroups H of B_n, whence $\mathcal{PB}(B_n)$ includes a unit α_n corresponding to $\varepsilon_n : B_n \to \mathbb{C}$. There also is a unit β_n of $\Omega(B_n, \mathcal{U}_n)$ corresponding to a natural extension of $\mathrm{sgn}_n : S_n \to \mathbb{C}$ to B_n such that $\alpha_n\beta_n$ corresponds to the restriction of $\mathrm{sgn}_{2n} : S_{2n} \to \mathbb{C}$ to B_n. By the description of β_n in terms of the characters $1_H^{B_n}$ for $H \in \mathcal{Z}_n \cap \mathcal{U}_n$, we have

$$\beta_n \in \Omega(B_n, \widetilde{\mathcal{Z}}_n)^\times \cap (\Omega(B_n, \mathcal{U}_n)^\times - \mathcal{PB}(B_n)^\times),$$

which proves $\mathcal{PB}(B_n)^\times$ to be isomorphic to the four group.