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Groups whose degree graph has three independent vertices ☆

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Let G be a finite group, and let $\mathrm{cd}(G)$ denote the set of degrees of the irreducible complex characters of G. This paper is a contribution to the study of the *degree graph* of G, that is, the prime graph built on the set $\mathrm{cd}(G)$. Namely, we characterize finite groups whose degree graph has three independent vertices (i.e., three vertices that are pairwise non-adjacent). Our result turns out to be a generalization of several previously-known theorems concerning the structure of the degree graph.

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1. Introduction

Let G be a finite group. A well-established research field in Character Theory is the study of the set $\mathrm{cd}(G)$, whose elements are the degrees of the irreducible complex characters of G. In fact, many results in the literature show that even this relatively small set of positive integers encodes nontrivial information about the structure of G; in particular, there is a significant interplay between the group structure and the "arithmetical structure" of $\mathrm{cd}(G)$ (i.e., the way in which the numbers in this set decompose into prime factors).

An important tool that has been devised in order to analyze this aspect of $\mathrm{cd}(G)$, is the *degree graph* $\Delta(G)$ of G; this is the simple undirected graph whose vertex set $\mathrm{V}(G)$ consists of all the prime numbers that divide some element in $\mathrm{cd}(G)$, while a subset $\{p, q\}$ of $\mathrm{V}(G)$ belongs to the edge set $\mathrm{E}(G)$ if and only if pq divides an element in $\mathrm{cd}(G)$.

In the case when G is a finite *solvable* group, a fundamental theorem by P.P. Pálfy ([18]) establishes that the size of a set of independent vertices in $\Delta(G)$ is at most two; in other words, if π is a subset of $\mathrm{V}(G)$ such that $|\pi| \geq 3$, then there exist $p, q \in \pi$ such that $\{p, q\}$ belongs to $\mathrm{E}(G)$. Another equivalent formulation is the following: if G is solvable, then the complement graph of $\Delta(G)$ contains no triangles. As immediate consequences, the number of connected components of $\Delta(G)$ is at most 2; if $\Delta(G)$ is disconnected, then each connected component induces a complete subgraph of $\Delta(G)$, whereas, in the connected case, the diameter of $\Delta(G)$ is at most three. (We note that Pálfy's result has been recently extended by Z. Akhlaghi, C. Casolo and the authors in [1]: for a finite solvable group G, the complement graph $\Delta(G)$ does not contain any cycle of odd length, and it is therefore a bipartite graph.)

As for non-solvable groups, the situation is different; while the diameter of $\Delta(G)$ (in the connected case) turns out to be at most three for any finite group G ([13] and [14]), this graph may have sets of independent vertices whose size is larger than two. For instance, the degree graph of the alternating group $\mathrm{Alt}(5)$ is an empty triangle and, in general, $\Delta(\mathrm{PSL}_2(2^a))$ has three connected components for every $a \geq 2$ (see Proposition 2.6). Nevertheless, a universal bound for the size of a set of independent vertices in $\Delta(G)$ was established by A. Moretó and P.H. Tiep in [17], and this bound turns out to be in fact three. Therefore, three is also the maximum number of connected components of $\Delta(G)$.

The aim of the present paper is to investigate in which respect the structure of a finite group G is influenced by the existence of three independent vertices in $\Delta(G)$. The relevance of this aspect might be suggested by the fact that the finite groups whose degree graph has three connected components are completely characterized as direct products of the kind $\mathrm{PSL}_2(2^a) \times A$, where $a \geq 2$ and A is an abelian finite group (see [12]). The main result of the paper is the following.

Theorem A. *Let G be a finite group, and let $\pi \subseteq \mathrm{V}(G)$ be such that $|\pi| = 3$. Then π is an independent set of $\Delta(G)$ if and only if $\mathbf{O}^{\pi'}(G) = S \times A$, where A is abelian and*

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