Journal of Algebra 512 (2018) 81–91

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The quadratic type of the 2-principal indecomposable modules of the double covers of alternating groups

John Murray

Department of Mathematics & Statistics, National University of Ireland Maynooth, Co Kildare, Ireland

ARTICLE INFO

Article history:
Received 4 April 2018
Available online 17 July 2018
Communicated by Gunter Malle

MSC:
20C30
20C20

Keywords:
Alternating group
Principal indecomposable module
Dual module
Quadratic form
Characteristic 2
Involution

ABSTRACT

The principal indecomposable modules of the double cover $2.\mathcal{A}_n$ of the alternating group over a field of characteristic 2 are enumerated using the partitions of n into distinct parts. We determine which of these modules afford a non-degenerate $2.\mathcal{A}_n$-invariant quadratic form. Our criterion depends on the alternating sum and the number of odd parts of the corresponding partition.

1. Introduction

Recall that an element of a finite group G is said to be 2-regular if it has odd order and real if it is conjugate to its inverse. Moreover a real element is strongly real if it is

E-mail address: John.Murray@mu.ie.

https://doi.org/10.1016/j.jalgebra.2018.06.034

inverted by an involution and otherwise it is said to be weakly real. If k is a field, then a kG-module is said to have quadratic type if it affords a non-degenerate G-invariant k-valued quadratic form. The following is a recent result of R. Gow and the author [3]:

Proposition 1. *Suppose that k is an algebraically closed field of characteristic 2. Then for any finite group G, the number of isomorphism classes of quadratic type principal indecomposable kG-modules is equal to the number of strongly real 2-regular conjugacy classes of G.*

Our focus here is on the double cover $2.\mathcal{A}_n$ of the alternating group $\mathcal{A}_n$. All real 2-regular elements of $\mathcal{A}_n$ are strongly real. So every self-dual principal indecomposable $k\mathcal{A}_n$-module has quadratic type. On the other hand, $2.\mathcal{A}_n$ may have real 2-regular elements which are not strongly real. In this note we determine which principal indecomposable $k(2.\mathcal{A}_n)$-modules have quadratic type.

Let $\mathcal{S}_n$ be the symmetric group of degree n and let $\mathcal{D}(n)$ be the set of partitions of n which have distinct parts. In [6, 11.5] G. James constructed an irreducible $k\mathcal{S}_n$-module D^μ for each partition $\mu \in \mathcal{D}(n)$. Moreover, he showed that the D^μ are pairwise non-isomorphic, and every irreducible $k\mathcal{S}_n$-modules is isomorphic to some D^μ.

As $\mathcal{A}_n$ has index 2 in $\mathcal{S}_n$, Clifford theory shows that the restriction $D^\mu{\downarrow_{\mathcal{A}_n}}$ is either irreducible or splits into a direct sum of two non-isomorphic irreducible $k\mathcal{A}_n$-modules. Moreover, every irreducible $k\mathcal{A}_n$-module is a direct summand of some $D^\mu{\downarrow_{\mathcal{A}_n}}$.

D. Benson determined [1] which $D^\mu{\downarrow_{\mathcal{A}_n}}$ are reducible and we recently determined [8] when the irreducible direct summands of $D^\mu{\downarrow_{\mathcal{A}_n}}$ are self-dual (see below for details). Throughout this paper we use D^μ_A to denote an irreducible direct summand of $D^\mu{\downarrow_{\mathcal{A}_n}}$.

As the centre of $2.\mathcal{A}_n$ acts trivially on any irreducible module, D^μ_A can be considered as an irreducible $k(2.\mathcal{A}_n)$-module, and all irreducible $k(2.\mathcal{A}_n)$-modules arise in this way.

The alternating sum of a partition μ is $|\mu|_a := \sum(-1)^{j+1}\mu_j$. We use $\ell_o(\mu)$ to denote the number of odd parts in μ. So $|\mu|_a \equiv \ell_o(\mu) \pmod 2$ and $|\mu|_a \geq \ell_o(\mu)$, if μ has distinct parts. Our result is:

Theorem 2. *Let μ be a partition of n into distinct parts and let P^μ be the projective cover of the simple $k(2.\mathcal{A}_n)$-module D^μ_A. Then P^μ has quadratic type if and only if*

$$\frac{n-|\mu|_a}{2} \leq 4m \leq \frac{n-\ell_o(\mu)}{2}, \quad \textit{for some integer } m.$$

Note that P^μ is a principal indecomposable $k(2.\mathcal{A}_n)$-module, but is not a $k\mathcal{A}_n$-module. Throughout the paper all our modules are left modules.