Centrally essential group algebras

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PII: S0021-8693(18)30417-4
DOI: https://doi.org/10.1016/j.jalgebra.2018.07.009
Reference: YJABR 16783

To appear in: *Journal of Algebra*

Received date: 11 December 2017

Please cite this article in press as: V.T. Markov, A.A. Tuganbaev, Centrally essential group algebras, *J. Algebra* (2018), https://doi.org/10.1016/j.jalgebra.2018.07.009

Centrally Essential Group Algebras

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Abstract. A ring R with center C is said to be *centrally essential* if the module R_C is an essential extension of the module C_C. In the paper, we study groups whose group algebras over fields are centrally essential rings. We focus on the centrally essential modular group algebras of finite groups over fields of nonzero characteristic.

V.T.Markov is supported by the Russian Foundation for Basic Research, project 17-01-00895-A. A.A.Tuganbaev is supported by Russian Scientific Foundation, project 16-11-10013.

Key words: centrally essential ring, group ring, FC-group.

1. Introduction

All considered rings are associative unital rings. A ring R with center C is said to be *centrally essential* if the module R_C is an essential extension of the module C_C. This condition can be written also as follows:

$$\forall r \in R \setminus \{0\}, \exists c \in C \text{ such that } c \cdot r \in C \setminus \{0\}. \tag{1}$$

Any commutative ring is a trivial example of a centrally essential ring. However, there also exist noncommutative centrally essential rings; in particular, there are noncommutative centrally essential group rings of finite groups. For example, let $F = \mathrm{GF}(2)$ be the field of order 2 and $G = Q_8$ the quaternion group, i.e., G is the group with two generators a, b and defining relations $a^4 = 1$, $a^2 = b^2$ and $aba^{-1} = b^{-1}$; see [3, Section 4.4]. Then the group algebra FG is a noncommutative centrally essential ring consisting of 256 elements (this follows from Theorem 1.1 below).

The main result of the paper is Theorem 1.1.

Theorem 1.1. Let F be a field of characteristic $p > 0$.

1. If G is an arbitrary finite group, then the group algebra FG is a centrally essential ring if and only if $G = P \times H$, where P is the unique Sylow p-subgroup of the group G, the group H is commutative, and the ring FP is centrally essential.