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BOUQUET ALGEBRA OF TORIC IDEALS

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ABSTRACT. To any toric ideal I_A, encoded by an integer matrix A, we associate a matroid structure called *the bouquet graph* of A and introduce another toric ideal called *the bouquet ideal* of A. We show how these objects capture the essential combinatorial and algebraic information about I_A. Passing from the toric ideal to its bouquet ideal via the graph theoretic properties of the bouquet graph allows us to classify several cases. For example, on the one end of the spectrum, there are ideals that we call *stable*, for which bouquets capture the complexity of various generating sets as well as the minimal free resolution. On the other end of the spectrum lie toric ideals whose various bases (e.g., minimal generating sets, Gröbner, Graver bases) coincide. Apart from allowing for classification-type results, bouquets provide a new way to construct families of examples of toric ideals with various interesting properties, such as robustness, genericity, and unimodularity. The new bouquet framework can be used to provide a characterization of toric ideals whose Graver basis, the universal Gröbner basis, any reduced Gröbner basis and any minimal generating set coincide.

INTRODUCTION

Toric ideals appear prominently in polyhedral geometry, algebraic topology, and algebraic geometry. Naturally, most famous classes of toric ideals come equipped with a rich algebraic and homological structure. They also have a common combinatorial feature, namely, equality of various bases. For example, generic toric ideals are minimally generated by indispensable binomials, robust toric by the universal Gröbner basis, those that are Lawrence are minimally generated by the Graver basis, and circuits equal the Graver basis for unimodular toric ideals.

In light of this, the present manuscript offers a combinatorial classification of all toric ideals. This classification simultaneously reveals equality of various distinguished subsets of binomials, provides a unifying framework for studying combinatorial signatures of toric ideals, and introduces a technique to solve several related (open) problems from combinatorial commutative algebra, algebraic geometry, combinatorics and integer programming. Furthermore, it provides a technique to construct (infinitely many) examples of five important classes of toric ideals (which turned out to be a challenge so far for generic, robust and strongly robust) often exploited in proving several important results in our paper. Before stating our main

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