Journal of Algebra 511 (2018) 95–101

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A birational embedding of an algebraic curve into a projective plane with two Galois points

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ARTICLE INFO

Article history:
Received 23 October 2017
Available online 30 June 2018
Communicated by Kazuhiko Kurano

MSC:
14H50
14H05
14H37

Keywords:
Galois point
Plane curve
Galois group
Automorphism group

ABSTRACT

A criterion for the existence of a birational embedding of an algebraic curve into a projective plane with two Galois points is presented. Several novel examples of plane curves with two inner Galois points as an application are described.

1. Introduction

The notion of *Galois point* was introduced by Hisao Yoshihara in 1996, to investigate the function fields of algebraic curves ([5,9]). For about twenty years, many interesting results have been obtained by several authors (Yoshihara, Miura, Takahashi, Fukasawa, et al., see also [11]). One of the most interesting problems in the theory of Galois point is

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[1] The author was partially supported by JSPS KAKENHI Grant Number 16K05088.

https://doi.org/10.1016/j.jalgebra.2018.06.020

to determine the number of Galois points for any plane curve. For smooth plane curves, the number of Galois points is completely determined ([3,9]). On the other hand, there are not so many known examples of (singular) plane curves with two Galois points (see the Tables in [11]). It is important to find a condition for the existence of two Galois points.

Let C be a (reduced, irreducible) smooth projective curve over an algebraically closed field k of characteristic $p \geq 0$ with $k(C)$ as its function field. We consider a rational map φ from C to $\mathbb{P}^2$, which is birational onto its image. For a point $P \in \mathbb{P}^2$, if the function field extension $k(\varphi(C))/\pi_P^* k(\mathbb{P}^1)$ induced by the projection π_P is Galois, then P is called a Galois point for $\varphi(C)$. Furthermore, if a Galois point P is a smooth point of $\varphi(C)$ (resp. is contained in $\mathbb{P}^2 \setminus \varphi(C)$), then P is said to be inner (resp. outer). The associated Galois group at P is denoted by G_P.

The following proposition is presented after discussions with Takahashi [7], Terasoma [8] and Yoshihara [10].

Proposition 1. *Let C be a smooth projective curve. Assume that there exist two finite subgroups, G_1 and G_2, of the full automorphism group* $\mathrm{Aut}(C)$ *such that $G_1 \cap G_2 = \{1\}$ and $C/G_i \cong \mathbb{P}^1$ for $i = 1, 2$. Let f and g be generators of function fields of C/G_1 and C/G_2, respectively. Then, the rational map*

$$\varphi : C \dashrightarrow \mathbb{P}^2; \ (f : g : 1)$$

is birational onto its image, and two points $P_1 = (0 : 1 : 0)$ and $P_2 = (1 : 0 : 0)$ are Galois points for $\varphi(C)$.

For both points P_1 and P_2 to be inner, or outer, we need additional conditions. In this article, we present the following criterion.

Theorem 1. *Let C be a smooth projective curve and let G_1 and G_2 be different finite subgroups of* $\mathrm{Aut}(C)$. *Then, there exist a morphism $\varphi : C \to \mathbb{P}^2$ and different inner Galois points $\varphi(P_1)$ and $\varphi(P_2) \in \varphi(C)$ such that φ is birational onto its image and $G_{\varphi(P_i)} = G_i$ for $i = 1, 2$, if and only if the following conditions are satisfied.*

(a) *$C/G_1 \cong \mathbb{P}^1$ and $C/G_2 \cong \mathbb{P}^1$.*
(b) *$G_1 \cap G_2 = \{1\}$.*
(c) *There exist two different points P_1 and $P_2 \in C$ such that*

$$P_1 + \sum_{\sigma \in G_1} \sigma(P_2) = P_2 + \sum_{\tau \in G_2} \tau(P_1)$$

as divisors.