Journal of Algebra 511 (2018) 102–113

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Witt group of a discretely valued field

Jón Kr. Arason

ARTICLE INFO

Article history:
Received 30 January 2018
Available online 30 June 2018
Communicated by Luchezar L. Avramov

Keywords:
Quadratic forms
Witt groups
Discrete valuations

ABSTRACT

This note provides an extension of a classical result of Springer in the Algebraic Theory of Quadratic Forms for discretely valued fields to the dyadic case.

In this note we shall study the relationship between the Witt group of quadratic forms over a discretely valued field and its residue class field.

The non-dyadic case is well known. The dyadic case, for more general valuations, has been studied in various papers. Examples are [4] and [2]. Here we restrict to discrete valuations of rank one and get more definitive results. Furthermore, our results are expressed in a new way.

For a commutative ring R we denote by $W(R)$ the Witt ring of symmetric bilinear forms and by $W_q(R)$ the Witt group of quadratic forms over R. We recall that $W_q(R)$ is a $W(R)$-module in a natural way. For exact definitions and general facts consult, for example, Baeza's book [3].

If 2 is invertible in R then $W_q(R)$ can be identified with the additive group of $W(R)$. We do it by letting the quadratic form $x \mapsto b(x,x)$ correspond to a given symmetric bilinear form b. (This differs from the choice in [3], (I.1.6).)

E-mail address: jka@hi.is.

https://doi.org/10.1016/j.jalgebra.2018.05.036

In what follows K is a field and ν is a valuation of K with value group $\mathbf{Z}$. We denote by $\mathbf{o}$ the valuation ring of ν, by $\mathbf{m}$ the maximal ideal of $\mathbf{o}$, and by $\mathbf{k} = \mathbf{o}/\mathbf{m}$ the residue class field.

We choose a prime element π in $\mathbf{o}$.

A well known generalized version of a theorem of Springer says that there is a short exact sequence $0 \to W(\mathbf{o}) \to W(K) \to W(\mathbf{k}) \to 0$ of additive groups, where the former morphism is the natural one and the latter one only depends on the choice of π. (See [6], remark after (IV.1.3).)

If ν is not dyadic then one can, using our identifications, think of this as an exact sequence $0 \to W_q(\mathbf{o}) \to W_q(K) \to W_q(\mathbf{k}) \to 0$ of Witt groups. We shall see that in general there is a natural subgroup $W_q(K)_{tame}$ of $W_q(K)$ containing $W_q(\mathbf{o})$ and a short exact sequence

$$0 \to W_q(\mathbf{o}) \to W_q(K)_{tame} \to W_q(\mathbf{k}) \to 0$$

of groups, where the former morphism is the inclusion and the latter one is a second residue class morphism corresponding to π. In the non-dyadic case $W_q(K)_{tame} = W_q(K)$ and the sequence is the one already mentioned.

It remains to describe $W_q(K)/W_q(K)_{tame}$ in the dyadic case. We note that there are two subcases depending on the characteristic of K: The mixed characteristic case, where $\mathrm{char}(K) = 0$, and the characteristic 2 case, where $\mathrm{char}(K) = 2$.

We shall see that there is a natural ascending filtration

$$W_q(K)_0 \subseteq W_q(K)_1 \subseteq W_q(K)_2 \subseteq \cdots$$

of $W_q(K)$ by subgroups $W_q(K)_n$ with $W_q(K)_0 = W_q(K)_{tame}$. This filtration is finite in the mixed characteristic case but infinite in the characteristic 2 case. Moreover, we shall describe the quotients $W_q(K)_n/W_q(K)_{n-1}$ for $n \geq 1$ in terms of $\mathbf{k}$.

More precisely, our main results are as follows.

Theorem 1. *Let $W_q(K)_{tame} = W(K)W_q(\mathbf{o})$. Then there is a short exact sequence*

$$0 \to W_q(\mathbf{o}) \to W_q(K)_{tame} \to W_q(\mathbf{k}) \to 0$$

of groups, where the former morphism is the inclusion but the latter one depends on the choice of π.

Theorem 2. *Assume that $\mathrm{char}(K) = 0$ but $\mathrm{char}(\mathbf{k}) = 2$. Write $e = \nu(2)$.*

There is a natural ascending filtration

$$W(K)_{tame} = W_q(K)_0 \subseteq W_q(K)_1 \subseteq \cdots \subseteq W_q(K)_{2e} = W_q(K)$$

of the group $W_q(K)$.