Journal of Algebra 511 (2018) 249–269

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Cosupport computations for finitely generated modules over commutative noetherian rings

Peder Thompson

Department of Mathematics and Statistics, Texas Tech University, Broadway and Boston, Lubbock, TX 79409, United States of America

ARTICLE INFO

Article history:
Received 23 February 2017
Available online 30 June 2018
Communicated by Luchezar L. Avramov

MSC:
13D02
13D07
13D09
13C11
13E05

Keywords:
Cosupport
Minimal complex
Cotorsion flat module
Countable ring

ABSTRACT

We show that the cosupport of a commutative noetherian ring is precisely the set of primes appearing in a minimal pure-injective resolution of the ring. As an application of this, we prove that every countable commutative noetherian ring has full cosupport. We also settle the comparison of cosupport and support of finitely generated modules over any commutative noetherian ring of finite Krull dimension. Finally, we give an example showing that the cosupport of a finitely generated module need not be a closed subset of Spec R, providing a negative answer to a question of Sather-Wagstaff and Wicklein [29].

Introduction

The theory of cosupport, recently developed by Benson, Iyengar, and Krause [7] in the context of triangulated categories, was partially motivated by work of Neeman [25], who classified the colocalizing subcategories of the derived category of a commutative

E-mail address: peder.thompson@ttu.edu.

https://doi.org/10.1016/j.jalgebra.2018.06.014

noetherian ring. Despite the many ways in which cosupport is dual to the more established notion of support introduced by Foxby [15,6], cosupport seems to be more elusive, even in the setting of a commutative noetherian ring. Indeed, the supply of finitely generated modules for which cosupport computations exist is limited. One purpose of this paper is to provide such computations.

We first show that for a finitely generated module over a commutative noetherian ring of finite Krull dimension, its cosupport is the intersection of its support and the cosupport of the ring, which places emphasis on computing the cosupport of the ring itself. With this in mind, we prove that countable commutative noetherian rings have full cosupport, and hence cosupport and support coincide for finitely generated modules over such rings having finite Krull dimension. We also give new examples of uncountable rings that have full cosupport. Finally, we present an example of a ring whose cosupport is not closed, unlike support, yielding a negative answer to a question posed by Sather-Wagstaff and Wicklein [29].

One method to determine the support of a module is to identify primes appearing in its minimal injective resolution, as done by Foxby [15], using the decomposition of injective modules described by Matlis [22]. Our systematic approach to computing cosupport is to appeal to the parallel decomposition of cotorsion flat modules due to Enochs [11] and use minimal cotorsion flat resolutions studied in [31].

Our goal is to better understand cosupport in the setting of a commutative noetherian ring. Over such a ring R, the cosupport of a complex M is denoted $\operatorname{cosupp}_R M$. This is the set of prime ideals $\mathfrak{p}$ such that $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \mathbf{L}\Lambda^\mathfrak{p} M)$ is not acyclic, where $\mathbf{L}\Lambda^\mathfrak{p}(-)$ is left derived $\mathfrak{p}$-adic completion; see Section 1 for details. Prompted by the fact that if M is a finitely generated $\mathbb{Z}$-module, then there is an equality $\operatorname{cosupp}_\mathbb{Z} M = \operatorname{supp}_\mathbb{Z} M$ [7, Proposition 4.18], we investigate to what extent cosupport and support agree for finitely generated modules. The cosupport of finitely generated modules over a 1-dimensional domain having a dualizing complex is known [29, Theorem 6.11]; this is recovered by part (2) of the following. Part (3) gives an affirmative answer to a question in [29].

Theorem 1 *(cf. Theorem 4.13, Corollary 4.14). Let R be one of the following:*

(1) *A countable commutative noetherian ring;*
(2) *A finite ring extension of a 1-dimensional commutative noetherian domain that is not complete local;*
(3) *A finite ring extension of $k[x,y]_{(x,y)}$ for any field k.*

Then R has full cosupport, i.e., $\operatorname{cosupp}_R R = \operatorname{Spec} R$.

If R is a countable commutative noetherian ring having finite Krull dimension or a ring as in (2) or (3), and M is an R-complex with degreewise finitely generated cohomology, then $\operatorname{cosupp}_R M = \operatorname{supp}_R M$.