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Skew braces and the Galois correspondence for Hopf Galois structures

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Let L/K be a Galois extension of fields with Galois group Γ, and suppose L/K is also an H-Hopf Galois extension. Using the recently uncovered connection between Hopf Galois structures and skew left braces, we introduce a method to quantify the failure of surjectivity of the Galois correspondence from subHopf algebras of H to intermediate subfields of L/K, given by the Fundamental Theorem of Hopf Galois Theory. Suppose $L \otimes_K H = LN$ where $N \cong (G, \star)$. Then there exists a skew left brace $(G, \star, \circ)$ where $(G, \circ) \cong \Gamma$. We show that there is a bijective correspondence between the set of intermediate fields E between K and L that correspond to K-subHopf algebras of H and a set of sub-skew left braces of G that we call the $\circ$-stable subgroups of $(G, \star)$. Counting these subgroups and comparing that number with the number of subgroups of $\Gamma \cong (G, \circ)$ describes how far the Galois correspondence for the H-Hopf Galois structure is from being surjective. The method is illustrated by a variety of examples.

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1. Introduction

Chase and Sweedler [8] introduced the concept of a Hopf Galois extension of commutative rings as a generalization of a classical Galois extension of fields L/K with Galois group Γ. The idea is to view the Galois structure on L as an action by the group ring $K\Gamma$, a K-Hopf algebra, and then replace $K\Gamma$ by a general cocommutative K-Hopf algebra H. In that setting, the Fundamental Theorem of Galois Theory (FTGT) of Chase and Sweedler [8] states that if L/K is an H-Hopf Galois extension of fields for H a K-Hopf algebra, then there is an injection from the set of K-sub-Hopf algebras of H to the set of intermediate fields $K \subseteq E \subseteq L$, given by sending a K-subHopf algebra J to the fixed ring

$$L^J = \{x \in L | h(x) = \epsilon(h)x \text{ for all } h \text{ in } J\}$$

where $\epsilon : H \to K$ is the counit map. The *strong form* of the FTGT holds if the injection is also a surjection. For a classical Galois extension of fields with Galois group Γ, the FTGT holds in its strong form. But it is known from [17] that in general, the Galois correspondence need not be surjective. So for any given example it is of interest to determine how far surjectivity fails.

Let L/K be a Galois extension of fields with Galois group Γ, and suppose L/K has an H-Hopf Galois structure of type G. Then H corresponds to a regular subgroup of $\text{Perm}(\Gamma)$ isomorphic to G and normalized by $\lambda(\Gamma)$, the image of the left regular representation of Γ in $\text{Perm}(\Gamma)$.

In turn, from [9] and [3], the Hopf Galois structure corresponds to a regular subgroup of $\text{Hol}(G)$ isomorphic to Γ, where $\text{Hol}(G) \subset \text{Perm}(G)$ is the holomorph of G, that is, the normalizer in $\text{Perm}(G)$ of $\lambda(G)$.

In turn, by results of increasing generality, from [6] to [16] to [1], [18] and [20], a regular subgroup of $\text{Hol}(G)$ isomorphic to Γ corresponds to a skew left brace $(G, \star, \circ)$ where $(G, \star) \cong G$ and $(G, \circ) \cong \Gamma$.

In [10] the question of the image of the Galois correspondence was studied for a Hopf Galois structure of type $G \cong \Gamma$ on a Galois extension L/K with p-elementary abelian Galois group Γ. The method used the [6] correspondence between regular subgroups of $\text{Hol}(G)$ and commutative radical algebra structures on G. In that setting the intermediate fields in the image of the Galois correspondence correspond to left ideals of the radical algebra A with circle group isomorphic to Γ and additive group isomorphic to G. Subsequently, [12] obtained upper and lower bounds on the number of left ideals in that setting, hence upper and lower bounds on the proportion of all intermediate fields that are in the image of the Galois correspondence.

In this paper we generalize [10] to the general setting of a skew left brace. Our main result obtains a bijective correspondence between intermediate fields in the image of the Galois correspondence and what we call $\circ$-stable subgroups of the additive group $(G, \star)$ of the skew left brace $(G, \star, \circ)$ corresponding to the Hopf Galois structure on L/K.