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Generating mapping class groups with elements of fixed finite order

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We show that for $k \geq 6$ and g sufficiently large, the mapping class group of a surface of genus g can be generated by three elements of order k. We also show that this can be done with four elements of order 5 when g is at least 8.

1. Introduction

Let S_g be a closed, connected, and orientable surface of genus g. The mapping class group $\mathrm{Mod}(S_g)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S_g. In this paper, we construct small generating sets for $\mathrm{Mod}(S_g)$ where all of the generators have the same finite order.

Theorem 1.1. *Let $k \geq 6$ and $g \geq (k-1)^2 + 1$. Then $\mathrm{Mod}(S_g)$ is generated by three elements of order k. Also, $\mathrm{Mod}(S_g)$ is generated by four elements of order 5 when $g \geq 8$.*

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Theorem 1.1 follows from a stronger but more technical result that we prove as Theorem 4.1. Our generating sets for $\mathrm{Mod}(S_g)$ are constructed explicitly. In addition, the elements in any particular generating set are all conjugate to each other. Of course, attempting to construct generating sets consisting of elements of a fixed order k only makes sense if $\mathrm{Mod}(S_g)$ contains elements of order k in the first place. A construction of Tucker [19] guarantees an element of any fixed order k in $\mathrm{Mod}(S_g)$ whenever g is sufficiently large, as described in Section 2.

Later in the introduction we describe prior work by several authors on generating $\mathrm{Mod}(S_g)$ with elements of fixed finite orders 2, 3, 4, and 6. In each case, the authors show that the number of generators required is independent of g. Set alongside this prior work, a new phenomenon that emerges in our results is that the sizes of our generating sets for $\mathrm{Mod}(S_g)$ are not only independent of the genus of the surface, but they are also independent of the order of the elements.

Our generators are all finite-order elements that can be realized by rotations of S_g embedded in $\mathbb{R}^3$. There are values of g and k where there exist elements of order k in $\mathrm{Mod}(S_g)$, but where these cannot be realized as rotations of S_g embedded in $\mathbb{R}^3$. For instance, there are elements of order 7 in $\mathrm{Mod}(S_3)$ that cannot be realized in this way.

Problem 1.2. *Extend Theorem 4.1 to cases where elements of order k exist in $\mathrm{Mod}(S_g)$ but cannot be realized as rotations of S_g embedded in $\mathbb{R}^3$.*

We can also seek smaller generating sets for $\mathrm{Mod}(S_g)$ consisting of elements of order k. We note that any such sharpening of Theorem 4.1 would seem to demand a new approach. Our proofs hinge on applications of the lantern relation, and a lantern has only a limited number of symmetries.

Problem 1.3. *For fixed $k \geq 3$ and any $g \geq 3$ where elements of order k exist, can $\mathrm{Mod}(S_g)$ be generated by two elements of order k? What about three elements for orders 4 and 5?*

Background and prior results. The most commonly-used generating sets for $\mathrm{Mod}(S_g)$ consist of Dehn twists, which have infinite order. Dehn [4] showed that $2g(g-1)$ Dehn twists generate $\mathrm{Mod}(S_g)$, and Lickorish [14] showed that $3g+1$ Dehn twists suffice. Humphries [9] showed that only $2g+1$ Dehn twists are needed, and he also showed that no smaller set of Dehn twists can generate $\mathrm{Mod}(S_g)$. The curves for these Dehn twists are depicted in Fig. 1.1.

There have also been many investigations into constructing generating sets for $\mathrm{Mod}(S_g)$ that include or even consist entirely of periodic elements. For instance, Maclachlan [16] showed that $\mathrm{Mod}(S_g)$ is normally generated by a set of two periodic elements that have orders $2g+2$ and $4g+2$, and McCarthy and Papadopoulos [17] showed that $\mathrm{Mod}(S_g)$ is normally generated by a single involution (element of order 2) for $g \geq 3$. Korkmaz [12] showed that $\mathrm{Mod}(S_g)$ is generated by two elements of order $4g+2$ for $g \geq 3$.