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Characterizing nilpotent Leibniz algebras by a new bound on their second homologies

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Abstract

Let L be a finite dimensional nilpotent Leibniz algebra such that $\dim(L) = n$ and $\dim(L^2) = m \neq 0$. In this paper, we prove $\dim(HL_2(L)) \leq (n+m-2)(n-m)-m+2$, where $HL_2(L)$ is the second Leibniz homology of L. As a consequence, for a non-abelian nilpotent Leibniz algebra L, we find that $s(L) = (n-1)^2 + 1 - \dim(HL_2(L)) \geq 0$. Furthermore, we determine all finite dimensional nilpotent Leibniz algebras with $s(L)$ less than or equal to three.

Keywords: nilpotent Leibniz algebras; Leibniz homology.
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1 Introduction and preliminary results

Leibniz algebras were introduced by Loday in [12, 13] as a non-commutative generalization of Lie algebras. A left (respectively, right) Leibniz algebra L is a vector space equipped with a bilinear product $[-,-] : L \times L \to L$ such that all left (respectively, right) product operators are derivation. Of course, any Lie algebra is both left and right Leibniz algebra but the converse does not hold in general case. It is convenient to show that for a Leibniz algebra L, the space spanned by squares of elements, $Leib(L) = span\{[x,x] | x \in L\}$, is an abelian ideal of L contained in the left center of L and L is a Lie algebra if and only if $Leib(L) = 0$. In this paper, we employ the notation of [8] and consider finite dimensional, left Leibniz algebras over an algebraically closed field $\mathbb{F}$ of characteristic different than 2.

For a Leibniz algebra L, the Leibniz homology of L with trivial coefficients, denoted by $HL_*(L)$, is the homology of the Loday complex $\left(\bigoplus_{n\geq 0} L^{\otimes n}, \partial_n\right)$, where the boundary map ∂ is given by

$$\partial_n(x_1 \otimes \cdots \otimes x_n) = \sum_{1\leq i<j\leq n} (-1)^j x_1 \otimes \cdots x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n, \qquad (n \geq 2)$$

and $\partial_1(x_1) = 0$, for any $x_1, \cdots, x_n \in L$. Therefore, $HL_1(L) = L/L^2$, $HL_0(L) = \mathbb{F}$ and if L is an abelian Lie algebra then $HL_n(L) = L^{\otimes n}$ for any $n \geq 1$.

Among all homology degrees of a Leibniz algebra, the second one is much more involved with the structural properties of a Leibniz algebra such as capability or perfectness, see for instance [2]. Also, there are several technical points of view and interpretations to the second homology group of a Leibniz algebra that we mention the most important ones.
Let $0 \to A \to K \to L \to 0$ be the maximal stem extension of the finite dimensional Leibniz algebra L, i.e. an exact sequence such that A is a central ideal of K that contained in K^2

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