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ON THE VARIETY OF 1-DIMENSIONAL REPRESENTATIONS OF FINITE W-ALGEBRAS IN LOW RANK

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Abstract. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and let $e \in \mathfrak{g}$ be nilpotent. We consider the finite W-algebra $U(\mathfrak{g}, e)$ associated to e and the problem of determining the variety $\mathcal{E}(\mathfrak{g}, e)$ of 1-dimensional representations of $U(\mathfrak{g}, e)$. For $\mathfrak{g}$ of low rank, we report on computer calculations that have been used to determine the structure of $\mathcal{E}(\mathfrak{g}, e)$, and the action of the component group Γ_e of the centralizer of e on $\mathcal{E}(\mathfrak{g}, e)$. As a consequence, we provide two examples where the nilpotent orbit of e is induced, but there is a 1-dimensional Γ_e-stable $U(\mathfrak{g}, e)$-module which is not induced via Losev's parabolic induction functor. In turn this gives examples where there is a "non-induced" multiplicity free primitive ideal of $U(\mathfrak{g})$.

1. Introduction

Let G be a simple algebraic group over $\mathbb{C}$, let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G, and let $e \in \mathfrak{g}$ be nilpotent. We write $U(\mathfrak{g}, e)$ for the finite W-algebra associated to $\mathfrak{g}$ and e. Finite W-algebras were introduced into the mathematical literature by Premet in [Pr1] in 2002, and have subsequently attracted a lot of research interest, we refer to [Lo2] for a survey up to 2010. The problem of understanding the 1-dimensional representations of $U(\mathfrak{g}, e)$ has been of particular interest due the relationship with completely prime and multiplicity free primitive ideals in $U(\mathfrak{g})$, and consequently quantizations of the algebra of regular functions on the nilpotent orbit of e; see for example [Pr4] and [Lo5], and the references therein. This paper makes a contribution by giving an explicit description of the variety of 1-dimensional representations of $U(\mathfrak{g}, e)$ for $\mathfrak{g}$ of low rank.

We introduce some notation required to discuss the background to and the contents of this paper further. Let I_c be the two-sided ideal of $U(\mathfrak{g}, e)$ generated by the commutators $uv - vu$ for $u, v \in U(\mathfrak{g}, e)$, and let $U(\mathfrak{g}, e)^{\mathrm{ab}} := U(\mathfrak{g}, e)/I_c$. The maximal spectrum $\mathcal{E} = \mathcal{E}(\mathfrak{g}, e)$ of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ parameterizes the 1-dimensional representations of $U(\mathfrak{g}, e)$. As explained in [PT, §5.1], there is an action of the component group $\Gamma = \Gamma_e$ of the centralizer of e in G on $U(\mathfrak{g}, e)^{\mathrm{ab}}$ and thus on $\mathcal{E}$. The fixed point variety of Γ in $\mathcal{E}$ is denoted by $\mathcal{E}^\Gamma$ and is identified with the maximal spectrum of $U(\mathfrak{g}, e)^{\mathrm{ab}}_\Gamma := U(\mathfrak{g}, e)^{\mathrm{ab}}/I_\Gamma$, where I_Γ is the two sided ideal of $U(\mathfrak{g}, e)^{\mathrm{ab}}$ generated by all elements of the form $u - \gamma \cdot u$ for $u \in U(\mathfrak{g}, e)^{\mathrm{ab}}$ and $\gamma \in \Gamma$. We let $\mathfrak{g}^e$ denote the centralizer of e in $\mathfrak{g}$, and note that there is an action of Γ on $\mathfrak{g}^e/[\mathfrak{g}^e, \mathfrak{g}^e]$ as explained in [PT, §5.1]. Let $c(e) := \dim(\mathfrak{g}^e/[\mathfrak{g}^e, \mathfrak{g}^e])$ and $c_\Gamma(e) := \dim((\mathfrak{g}^e/[\mathfrak{g}^e, \mathfrak{g}^e])^\Gamma)$. We write $\mathcal{O}_e \subseteq \mathfrak{g}$ for the nilpotent orbit of e.

We briefly give an overview of previous research on 1-dimensional representations of $U(\mathfrak{g}, e)$, and refer to the introductions to [PT] and [Pr4] for a more detailed account.

In [Pr2, Conjecture 3.1], Premet predicted that $\mathcal{E} \neq \varnothing$ for all $\mathfrak{g}$ and e, i.e. that there exists a 1-dimensional representation of $U(\mathfrak{g}, e)$. For $\mathfrak{g}$ of classical type, Losev proved the existence of 1-dimensional representations of $U(\mathfrak{g}, e)$ in [Lo1, Theorem 1.2.3]. In [Pr3, Theorem 1.1],

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