ARTICLE IN PRESS YJABR:16669

Journal of Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Stabilization bounds for linear finite dynamical systems

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A R T I C L E I N F O

Article history:
Received 7 September 2017
Available online xxxx
Communicated by Reinhard Laubenbacher

MSC:
05C50
13M99
13P99
37P25

Keywords:
Finite rings
Linear finite dynamical systems
Fixed point systems
Height

A B S T R A C T

A common problem to all applications of linear finite dynamical systems is analyzing the dynamics without enumerating every possible state transition. Of particular interest is the long term dynamical behaviour. In this paper, we study the number of iterations needed for a system to settle on a fixed set of elements. As our main result, we present two upper bounds on iterations needed, and each one may be readily applied to a fixed point system test or a computation of limit cycles. The bounds are based on submodule properties of iterated images and reduced systems modulo a prime. We also provide examples where our bounds are optimal.

1. Introduction

A *finite dynamical system* is an ordered pair (X, f), which consists of a finite set X together with an iterating function f over X. That is, elements of X evolve by repeated application of f. Thus the study of a finite dynamical system is a study of sequences of the form

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https://doi.org/10.1016/j.jalgebra.2018.04.009
0021-8693/

Please cite this article in press as: B. Lindenberg, Stabilization bounds for linear finite dynamical systems, J. Algebra (2018), https://doi.org/10.1016/j.jalgebra.2018.04.009

$$(x, f(x), f^2(x), \cdots), \tag{1}$$

where f is called the *defining function*. In mathematical modelling, finite dynamical systems emerge in many different areas. Applications can be found in, for example, computational physics, electrical engineering, artificial intelligence, gene regulatory networks and cellular automata [1–4]. Hence, f often represents *state transitions* or the *passing of time*.

In this context, a *linear finite dynamical system* (M, f) is a system where f is a linear map and M is a *finite R-module*. That is, M is a generalized vector space with scalars in a finite commutative ring R, and is generated by a finite set of *base elements*. Therefore, f can be represented by a matrix A over R, which describes the mapping of the base elements.

An important problem in applications of finite dynamical systems, is the determination of long term behaviour without enumeration of all possible state transitions. The size of the underlying set X may lead to iteration over all elements being computationally intractable. As an example, a finite dynamical system in the form of a Boolean modelling framework is demonstrated in [5]. There the framework models a gene regulatory network of 60 nodes, such that $|X| = 2^{60}$. Other frameworks for gene regulatory networks may have more than two states for each node, leading to a further increase in the size of the underlying set [4].

When a linear finite dynamical system is given by $(\mathbb{Z}_p^m, A)$, where p is prime, the properties and long term behaviour is well understood [1,2,4]. Although when dealing with more general linear systems $(\mathbb{Z}_n^m, A)$, one faces difficulties due to the lack of unique factorization [6,7].

1.1. Fixed point systems

Let (X, f) be a finite dynamical system and consider an iterated sequence of the form given in (1). It is possible that after a certain number of iterations we reach a *fixed point* x_0, such that $f(x_0) = x_0$. If every element in X eventually iterates to a fixed point, not necessarily unique, then the system is called a *fixed point system*. Otherwise elements of X will eventually converge within a subset of X, which consists of fixed points and *cycles*, where cycles are closed iteration loops of more than one element. An extensive study of cycles of linear finite dynamical systems can be found in [7].

A *fixed point system criterion* for $(\mathbb{F}_q^n, A)$, where $\mathbb{F}_q$ is the finite field with q elements, is given by Bollman et al. in [4]. It is based on the *minimal polynomial* of A, i.e., the polynomial of least degree in $\mathbb{F}_q[x]$ which annihilates A. As an alternative, G. Xu and Y.M. Zou presented another fixed point system criterion in [6]. Given (R^m, A), where R is a finite commutative ring, they showed that if the size of R is a composite integer n, then the system is a fixed point system if and only if

$$A^{k+1} = A^k, \tag{2}$$