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# Half-axes in power associative algebras

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A B S T R A C T

Let $A$ be a commutative, non-associative algebra over a field $\mathbb{F}$ of characteristic $\neq 2$. A half-axis in $A$ is an idempotent $e \in A$ such that $e$ satisfies the Peirce multiplication rules in a Jordan algebra, and, in addition, the 1-eigenspace of $\mathrm{ad}_e$ (multiplication by $e$) is one dimensional.
In this paper we consider the identities
$(*)\quad x^2x^2 = x^4$ and $x^3x^2 = xx^4$.
We show that if identities $(*)$ hold strictly in $A$, then one gets (very) interesting identities between elements in the eigenspaces of $\mathrm{ad}_e$ (note that if $|\mathbb{F}| > 3$ and the identities $(*)$ hold in $A$, then they hold strictly in $A$). Furthermore we prove that if $A$ is a primitive axial algebra of Jordan type half (i.e., $A$ is generated by half-axes), and the identities $(*)$ hold strictly in $A$, then $A$ is a Jordan algebra.


## 1. Introduction

Throughout this paper $\mathbb{F}$ is a field of characteristic not 2 and $A$ is a commutative non-associative algebra over $\mathbb{F}$. Given an element $x \in A$ and a scalar $\lambda \in \mathbb{F}$, we denoted:

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$$A_\lambda(x) := \{y \in A \mid yx = \lambda y\}.$$

(We allow $A_\lambda(x) = 0$.)

**Definition 1.1.** Let $e \in A$, and set $Z := A_0(e)$ and $U := A_{1/2}(e)$. We say that $e$ is a *half-axis* if and only if

(1) $e^2 = e$ (so $e$ is an idempotent).
(2) $A_1(e) = \mathbb{F}e$.
(3) $A = \mathbb{F}e \oplus U \oplus Z$.
(4) $Z^2 \subseteq Z,\ U^2 \subseteq \mathbb{F}e + Z$ and $UZ \subseteq U$.

Note that any idempotent $e$ in a Jordan algebra $J$ such that $J_1(e) = \mathbb{F}e$ is a half-axis.

Recall that $A$ is a *primitive axial algebra of Jordan type half* if $A$ is generated (as an algebra over $\mathbb{F}$) by half-axes.

We also need the following notation.

**Notation 1.2.** Let $e \in A$ be a half-axis, and let $x \in A$. Write $x = \alpha e + x_0 + x_{1/2}$, with $\alpha \in \mathbb{F}$ and $x_\lambda \in A_\lambda(e)$, for $\lambda \in \{0, 1/2\}$.

(1) We denote $\varphi_e(x) = \delta_x := \alpha$.
(2) We denote $z_x := x_0$. We call $z_x$ *the $Z$-part of $x$.*

Note that $ex = \delta_x e$, for $x \in A_1(e) + A_0(e)$.

Throughout this paper we shall use the technique of *linearization* of identities. More details about this technique are given in §2.

*1.3. Scalar extension and strict validity of identities*

For a field extension $\mathbb{K}/\mathbb{F}$, we denote by $A_{\mathbb{K}} := A \otimes_{\mathbb{F}} \mathbb{K}$ the scalar extension (or base change) of $A$ from $\mathbb{F}$ to $\mathbb{K}$, which is a commutative non-associative $\mathbb{K}$-algebra in a natural way. It is well known that Jordan algebras are invariant under base change (see e.g. [4, Linearization Proposition 1.8.5(2), p. 148]), so $A$ is a Jordan algebra over $\mathbb{F}$ if and only if $A_{\mathbb{K}}$ is one over $\mathbb{K}$. Moreover, since tensor products commute with direct sums, if $e \in A$ is a half-axis, then $e$ is a half-axis in $A_{\mathbb{K}}$. Since primitive axial algebras of Jordan type half are *spanned* by half-axes (see [1, Corollary 1.2, p. 81]), it follows that primitive axial algebras are stable under base change as well. *But power-associative algebras are not.* For this reason, the concept of strict power-associativity comes in: $A$ is called *strictly power-associative* if the scalar extensions $A_{\mathbb{K}}$ are power-associative, for all field extensions $\mathbb{K}/\mathbb{F}$. Similarly, an identity is said to *hold strictly* in $A$ if it is satisfied not only by $A$ but by all its scalar extensions.