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Space forms and group resolutions: The tetrahedral family

Rocco Chirivì *, Mauro Spreafico

Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Italy

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ABSTRACT

The orbit polytope for a finite group G acting linearly and freely on a sphere S is used to construct a cellularized fundamental domain for the action. A resolution of $\mathbb{Z}$ over G results from the associated G-equivariant cellularization of S. This technique is applied to the generalized binary tetrahedral group family; the homology groups, the cohomology rings and the Reidemeister torsions of the related spherical space forms are determined.

1. Introduction

If R is a ring and M an R-module, a resolution of M is an exact sequence of R-modules

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\epsilon} M \longrightarrow 0.$$

* Corresponding author.
E-mail addresses: rocco.chirivi@unisalento.it (R. Chirivì), mauro.spreafico@unisalento.it (M. Spreafico).

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Resolutions appear as fundamental objects both in algebra and in topology. In topology, where the ring R is usually the group ring $\mathbb{Z}G$ of the fundamental group G of some space, they represent a basic tool in dealing with the cohomology of groups as well as permit to compute the main algebraic topological invariants of a space. Unfortunately, to obtain an explicit resolution is in general a very difficult task. A standard technique is to use a simplicial or cellular decomposition of the space, or a G-equivariant decomposition of its universal covering. However an explicit decomposition is very hard but for the simplest examples of surfaces and lens spaces.

This approach has been particularly fruitful in the context of a G finite group acting freely on a sphere (see [14] for a list of these groups). These groups have been intensively studied in topology, since they appear as fundamental groups of the spherical space forms, manifolds whose universal covering is a sphere (see for example [6] and references therein). An explicit knowledge of a "reasonably simple" free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ would carry all interesting algebraic and geometric information; such a resolution for the simplest cases of the cyclic groups and the quaternionic groups has been classically known (see Cartan and Eilenberg [4, XII.7]). However, afterwards, this approach was somehow moved aside in favour to other techniques, mainly because of the intrinsic difficulty in obtaining suitable simple resolutions (see for example [20] for a survey).

Recently, refining the geometric approach introduced by M.M. Cohen (see [5]), the second author et al. (see [13] and [21]) succeeded to find such resolutions for all non abelian groups acting freely and linearly on S^3, except for the generalized binary tetrahedral groups. Indeed a direct approach to the construction of a G-equivariant cellular decomposition of the sphere, for G a generalized binary tetrahedral group, turns out to be almost impossible but for the first group of the family, i.e. the binary tetrahedral group (see [22]).

In this paper, given a finite group G freely acting on a sphere $\mathsf{S}^n \subseteq V$ by a linear representation $\rho : G \longrightarrow \mathsf{GL}(V)$, we construct a G-equivariant cellular decomposition in a uniform way. We start by choosing a point $v_0 \in \mathsf{S}^n$, consider the orbit $G \cdot v_0$ and its convex hull $\mathcal{P}$; this is a polytope, called the *orbit polytope*, on which faces the group G acts. The main idea is to use the orbit polytope to derive the cellular decomposition. A similar approach has been used in [9] and applied to a new proof of a resolution for finite reflection groups due to De Concini and Salvetti [7].

In our situation G acts freely on the faces of $\mathcal{P}$ and we prove that there exists a choice of representatives for the facets under this action, whose union projected on S^n is a fundamental domain.

The combinatorics of the faces of the polytope $\mathcal{P}$ depend on the choice of the point v_0. In order to simplify this combinatorics, finding a somehow natural choice for v_0, we locate an as large as possible cyclic subgroup H of G and take for v_0 an eigenvector for H in V. The restriction $\rho_H = \mathrm{Res}^G_H \rho$ of the representation ρ has the complex line Π_0 generated by v_0 as a summand and on Π_0 (a real plane) the H-orbit of v_0 is a polygon $\mathcal{P}_H$ with $|H|$ vertices.