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Fusion systems containing pearls ☆

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ABSTRACT

An $\mathcal{F}$-essential subgroup is called a *pearl* if it is either elementary abelian of order p^2 or non-abelian of order p^3. In this paper we start the investigation of fusion systems containing pearls: we determine a bound for the order of p-groups containing pearls and we classify the saturated fusion systems on p-groups containing pearls and having sectional rank at most 4.

0. Introduction

In finite group theory, the word *fusion* refers to the study of conjugacy maps between subgroups of a group. This concept has been investigated for over a century, probably starting with Burnside, and the modern way to solve problems involving fusion is via the theory of fusion systems. Given any finite group G, there is a natural construction of a saturated fusion system on one of its Sylow p-subgroups S: this is the category with objects the subgroups of S and with morphisms between subgroups P and Q of S

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given by the set $\mathrm{Hom}_G(P, Q)$ of all the restrictions of conjugacy maps by elements of G that map P into Q. In general, a saturated fusion system on a p-group S is a category whose objects are the subgroups of S and whose morphisms are the monomorphisms between subgroups which satisfy certain axioms, motivated by conjugacy relations and first formulated in the nineties by the representation theorist Puig (cf. [29]). There are saturated fusion systems which do not arise as fusion systems of a finite group G on one of its Sylow p-subgroups; these fusion systems are called *exotic*. The Solomon fusion systems $\mathrm{Sol}(p^a)$ (predicted by Benson and studied by Levi and Oliver in [22]) form the only known family of exotic simple fusion systems on 2-groups. In contrast, for odd primes p, there is a plethora of exotic fusion systems (see for example [7,24,26,30]). The classification results we prove in this paper lead us to the description of a new exotic fusion system on a 7-group of order 7^5.

The starting point toward the classification of saturated fusion systems is given by the Alperin–Goldschmidt Fusion Theorem [2, Theorem 1.19], that guarantees that a saturated fusion system $\mathcal{F}$ on a p-group S is completely determined by the $\mathcal{F}$-automorphism group of S and by the $\mathcal{F}$-automorphism group of certain subgroups of S, that are called for this reason *$\mathcal{F}$-essential*. More precisely, if $\mathcal{F}$ is a saturated fusion system on a p-group S, then a subgroup E of S is $\mathcal{F}$-essential if

- E is $\mathcal{F}$-centric: $\mathrm{C}_S(E\alpha) \leq E\alpha$ for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(E, S)$;
- E is fully normalized in $\mathcal{F}$: $|\mathrm{N}_S(E)| \geq |\mathrm{N}_S(E\alpha)|$ for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(E, S)$;
- $\mathrm{Out}_{\mathcal{F}}(E) = \mathrm{Aut}_{\mathcal{F}}(E)/\mathrm{Inn}(E)$ contains a strongly p-embedded subgroup.

The smallest candidate for an $\mathcal{F}$-essential subgroup is a group isomorphic to the direct product $\mathrm{C}_p \times \mathrm{C}_p$, since the outer automorphism group of a cyclic group does not have strongly p-embedded subgroups. The smallest candidate for a non-abelian $\mathcal{F}$-essential subgroup is a non-abelian group of order p^3 (that is isomorphic to the group p_+^{1+2} when p is odd). The purpose of this paper is to start the investigation of saturated fusion systems $\mathcal{F}$ containing these small $\mathcal{F}$-essential subgroups.

Definition 1. An *$\mathcal{F}$-pearl* is an $\mathcal{F}$-essential subgroup of S that is either elementary abelian of order p^2 or non-abelian of order p^3.

When this does not lead to confusion, we will omit the $\mathcal{F}$ in front of the name pearl.

Fusion systems containing pearls are far from being rare. Pearls appear in the study of saturated fusion systems on p-groups having a maximal subgroup that is abelian ([7,24,26]). Pearls are also contained in many of the saturated fusion systems on a Sylow p-subgroup of the group $\mathrm{G}_2(p)$, as proved in [28]; in particular the fusion system of the Monster group on one of its Sylow 7-subgroups contains an abelian pearl. Many saturated fusion systems on p-groups of small sectional rank contain pearls as well, when p is odd. The rank of a finite group G is the minimum size of a generating set for G and a finite p-group S has sectional rank k if every elementary abelian quotient P/Q