Journal of Algebra 510 (2018) 180–204

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Schurity and separability of quasiregular coherent configurations

Mitsugu Hirasaka[a,*,1], Kijung Kim[a,2], Ilia Ponomarenko[b,3]

[a] *Department of Mathematics, Pusan National University, Geumjung-gu, Busan 46241, Busan, Republic of Korea*
[b] *St.Petersburg Department of the Steklov Mathematical Institute, St.Petersburg, Russia*

ARTICLE INFO

Article history:
Received 4 January 2018
Available online xxxx
Communicated by Gernot Stroth

Keywords:
Schurity
Separability
Coherent configurations
Association schemes
Quasi-regular

ABSTRACT

A permutation group is said to be quasiregular if each of its transitive constituents is regular, and a quasiregular coherent configuration can be thought as a combinatorial analog of such a group: the transitive constituents are replaced by the homogeneous components. In this paper, we are interested in the question when the configuration is schurian, i.e., formed by the orbitals of a permutation group, or/and separable, i.e., uniquely determined by the intersection numbers. In these terms, an old result of Hanna Neumann is, in a sense, dual to the statement that the quasiregular coherent configurations with cyclic homogeneous components are schurian. In the present paper, we (a) establish the duality in a precise form and (b) generalize the latter result by proving that a quasiregular coherent configuration is schurian and separable

* Corresponding author.
E-mail addresses: hirasaka@pusan.ac.kr (M. Hirasaka), knukkj@pusan.ac.kr (K. Kim), inp@pdmi.ras.ru (I. Ponomarenko).

[1] The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (NRF-2016R1A2B4013474).
[2] The second author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education (NRF-2017R1D1A3B03031349).
[3] The work of the third author was supported by the RAS Program of the Presidium of the Russian Academy of Sciences No. 1 “Fundamental Mathematics and its Applications” under grant PRAS-18-01.

https://doi.org/10.1016/j.jalgebra.2018.05.027

if the groups associated with the homogeneous components have distributive lattices of normal subgroups.

1. Introduction

A permutation group is said to be *quasiregular* if each of its transitive constituents is regular [1, p. 53]. This concept has an obvious analog in the theory of coherent configuration: a coherent configuration is said to be quasiregular if each of its homogeneous components is regular (for exact definitions concerning coherent configurations, see Section 2). In fact, there is a one-to-one correspondence between the regular groups (which are exactly transitive quasiregular groups) and regular coherent configurations [11]. However, no such correspondence exists for intransitive quasiregular groups: the reason is that not every quasiregular coherent configuration is *schurian*, i.e., formed by the orbitals of a permutation group. The first example of a non-schurian quasiregular coherent configuration was constructed by Sergei Evdokimov in the end of 1990s (but was never published). One motivation for the present paper is to find conditions for a quasiregular coherent configuration to be schurian.

A second motivation comes from a paper of Hanna Neumann [10],[4] on amalgams of finite cyclic groups. Roughly speaking, the question is whether a family $\mathfrak{G}$ of groups with prescribed pairwise intersections *admits an amalgam*, i.e., can be isomorphically embedded into a certain group. The main result of [10] states that if $\mathfrak{G}$ consists of finite cyclic groups, then this is true if some natural necessary conditions concerning the pairwise intersections are satisfied. These conditions make sense not only for cyclic groups; any family $\mathfrak{G}$ satisfying these conditions is called a system of linked subgroups based on $\mathfrak{G}$ (the exact definition is given in Section 3). Not every system of linked subgroups based on a family $\mathfrak{G}$ of abelian groups admits an amalgam. A reason for this is revealed in the theorem below establishing a close relationship between the systems of linked subgroups based on $\mathfrak{G}$ and quasiregular coherent configurations of type $\mathfrak{G}$, i.e., those coherent configurations the homogeneous components of which are exactly the groups from $\mathfrak{G}$.

Theorem 1.1. *Let $\mathfrak{G}$ be a family of finite abelian groups. Then the quasiregular coherent configurations of type $\mathfrak{G}$ are in one-to-one correspondence with the systems of linked subgroups based on $\mathfrak{G}$. Moreover, a coherent configuration of type $\mathfrak{G}$ is schurian if and only if the corresponding system of linked subgroups admits an amalgam.*

The proof of Theorem 1.1 is presented in Section 5 and uses a quite general concept of a system of linked sections based on a family $\mathfrak{G}$ of groups (see Section 3). Two special

[4] The third author is expressed gratitude to Mikhail Muzychuk, who drew his attention to paper [10].