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Virtually free finite-normal-subgroup-free groups are strongly verbally closed

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A B S T R A C T

Any virtually free group H containing no non-trivial finite normal subgroup (e.g., the infinite dihedral group) is a retract of any finitely generated group containing H as a verbally closed subgroup.

0. Introduction

A subgroup H of a group G is called *verbally closed* [6] (see also [7], [2], [4], [1], [5]) if any equation of the form

$$w(x_1, x_2, \dots) = h, \quad \text{where } w \text{ is an element of the free group } F(x_1, x_2, \dots) \text{ and } h \in H,$$

having a solution in G has a solution in H. If each finite system of equations with coefficients from H

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$$\{w_1(x_1, x_2, \dots) = 1, \dots, w_m(x_1, x_2, \dots) = 1\}, \quad \text{where } w_i \in H * F(x_1, x_2, \dots),$$

having a solution in G has a solution in H, then the subgroup H is called *algebraically closed* in G.

Surely, the algebraic closedness is stronger than the verbal closedness. However, these properties turn out to be equivalent in many cases. A group H is called *strongly verbally closed* [5] if it is algebraically closed in any group containing H as a verbally closed subgroup. (Thus, verbal closedness is a property of a subgroup, while the strong verbal closedness is a property of an abstract group.) For example, the following groups are strongly verbally closed:

- all abelian groups [5];
- all free groups [1];
- the fundamental groups of all connected surfaces, except possibly the Klein bottle [5].

The main result of this paper can be stated as follows.

Theorem 1. *The following groups are strongly verbally closed:*

1) *all virtually free group containing no non-trivial finite normal subgroups;*
2) *all free products* $\mathop{*}\limits_{i \in I} H_i$, *where the set* I *is finite or infinite,* $|I| > 1$, *and* H_i *are nontrivial groups satisfying nontrivial laws.*

A large part of Theorem 1 was known earlier: in [5], Assertion 2) was proved for all non-dihedral groups under the additional condition that I is finite; in [1], the strong verbal closedness was proved for all infinite virtually free non-dihedral groups containing no infinite abelian noncyclic subgroups. Paper [1] contains also examples of virtually free groups that are not strongly verbally closed.

Actually, most of this paper is devoted to the proof of the following particular case of (both assertions of) Theorem 1:

the infinite dihedral group is strongly verbally closed.

For all non-dihedral groups, Theorem 1 is relatively easily derived from known facts (in Section 1).

The difficulty with the infinite dihedral group is that it is "too abelian" to apply sophisticated tools based on the Lee words [3] (see [6], [4], [1], [5]); on the other hand, it is "too nonabelian" to apply simple arguments (see [1], [5]). Of course, the dihedral group is metabelian and this is the basis of our approach. In Section 3, we give an "explicit" criterion for an infinite dihedral subgroup to be verbally (and algebraically) closed. This