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Pseudo-reductive and quasi-reductive groups over non-archimedean local fields

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Among connected linear algebraic groups, quasi-reductive groups generalize pseudo-reductive groups, which in turn form a useful relaxation of the notion of reductivity. We study quasi-reductive groups over non-archimedean local fields, focusing on aspects involving their locally compact topology.

For such groups we construct valuated root data (in the sense of Bruhat–Tits) and we make them act nicely on affine buildings. We prove that they admit Iwasawa and Cartan decompositions, and we construct small compact open subgroups with an Iwahori decomposition.

We also initiate the smooth representation theory of quasi-reductive groups. Among others, we show that their irreducible smooth representations are uniformly admissible, and that all these groups are of type I.

Finally we discuss how much of these results remains valid if we omit the connectedness assumption on our linear algebraic groups.

Contents

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Introduction

Pseudo-reductive and quasi-reductive groups generalize connected reductive groups. These classes of linear algebraic groups were already known to Tits [5] and Springer [25], who developed some basic theory. The recent work of Conrad, Gabber and Prasad [12–14] has revived the interest in these groups.

These sources are mainly concerned with the structure of pseudo-reductive groups over arbitrary or separably closed fields. Properties which involve the locally compact topology of pseudo-reductive or quasi-reductive groups over local fields have been investigated far less (apart from reductive groups of course). With this paper we try to narrow that gap.

Let $\mathcal{G}$ be a connected linear algebraic group defined over a field F. The F-unipotent radical $\mathcal{R}_{u,F}(\mathcal{G})$ of $\mathcal{G}$ is the largest connected normal unipotent F-subgroup of $\mathcal{G}$ (so it is contained in the usual unipotent radical of $\mathcal{G}$). By definition, $\mathcal{G}$ is pseudo-reductive as F-group if $\mathcal{R}_{u,F}(\mathcal{G}) = 1$.

Over a field of characteristic zero, every connected linear algebraic group $\mathcal{G}$ admits a Levi decomposition, that is, it can be written as the semidirect product of its unipotent radical and a reductive subgroup. But this is not always the case over fields F of positive characteristic p. Firstly, suitable Levi factors need not exist, even if F is algebraically closed [12, §A.6]. Secondly, the unipotent radical need not be defined over F. For example, suppose that F'/F is an inseparable field extension of degree p and that $\mathcal{G}'$ is a nontrivial connected reductive F'-group. Then restriction of scalars yields a pseudo-reductive F-group $R_{F'/F}(\mathcal{G}')$ which is not reductive [12, Proposition 1.1.10].

Nevertheless, $\mathcal{G}$ is always an extension of a pseudo-reductive F-group by a unipotent F-group. Namely, in the short exact sequence

$$1 \to \mathcal{R}_{u,F}(\mathcal{G}) \to \mathcal{G} \to \mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G}) \to 1 \tag{1}$$

the quotient group $\mathcal{G}/\mathcal{R}_{u,F}(\mathcal{G})$ is easily seen to be pseudo-reductive over F. Thus one can try to understand connected linear algebraic F-groups in terms of pseudo-reductive