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Odd orthogonal matrices and the non-injectivity of the Vaserstein symbol

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ABSTRACT

R.A. Rao–W. van der Kallen showed that the Vaserstein symbol $V_{\Gamma(S^3_{\mathbb{R}})}$ from the orbit space of unimodular rows of length three over the coordinate ring of the real three sphere $S^3_{\mathbb{R}}$ modulo elementary action to the elementary symplectic Witt group $W_E(\Gamma(S^3_{\mathbb{R}}))$ is not injective. Dhvanita R. Rao–Neena Gupta gave an uncountable family of singular real threefolds A_α for which the Vaserstein symbol V_{A_α} is not injective. In this paper, we give a countable family of **smooth** real birationally equivalent threefolds A_n for which the Vaserstein symbol V_{A_n} is not injective.

1. Introduction

In [8] L.N. Vaserstein described a beautiful Witt group structure on the orbit space of unimodular rows of length three $\mathrm{Um}_3(A)$ modulo the action of the elementary

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subgroup $\mathrm{E}_3(A)$, when A is a commutative noetherian ring of Krull dimension two. This association between the orbit space and the elementary symplectic Witt group $V_A : \mathrm{Um}_3(A)/\mathrm{E}_3(A) \longrightarrow \mathrm{W}_{\mathrm{E}}(A)$ is defined, for $v = (a, b, c)$, $w = (a', b', c')$, with $\langle v, w \rangle = 1$, by $[v] \longrightarrow [V(v, w)] \in W_{\mathrm{E}}(A)$, where

$$V(v,w) = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c' & -b' \\ -b & -c' & 0 & a' \\ -c & b' & -a' & 0 \end{pmatrix},$$

and the class $[V(v, w)]$ does not depend on the choice of w. This map can be studied in any dimension, and is now called the Vaserstein symbol. For more on this symbol we refer the reader to [8].

In [6] R.A. Rao–W. van der Kallen studied V_A when A is an affine threefold over a field k. They proved that if A is a smooth affine threefold over a field of cohomological dimension at most one, and of characteristic $\neq 2, 3$, then V_A is an isomorphism.

However, for the coordinate ring of polynomial functions on the real 3-sphere $\Gamma(S^3_{\mathbb{R}})$ they proved that $V_{\Gamma(S^3_{\mathbb{R}})}$ is not injective. This was the first example of a threefold for which Vaserstein symbol was shown to be non-injective.

The first named author with Neena Gupta studied the question of whether there are more examples of real affine threefolds for which the Vaserstein symbol is not injective. In [4], they presented an uncountable family of non-isomorphic affine threefolds over the real field for which the Vaserstein symbol is not injective.

The question then arose whether one could find an uncountable family of non-isomorphic smooth affine threefolds over the real field for which the Vaserstein symbol is not injective. In this article, we produce a countable family of such examples.

We begin by analysing the Rao–van der Kallen example afresh. We realized that the example is due to the existence of a 3×3 orthogonal matrix σ over $\Gamma(S^3_{\mathbb{R}})$ which had the property that its first row $e_1\sigma$, and also the first row of its square $e_1\sigma^2$ were not in the elementary orbit of $e_1 = (1, 0, 0)$. This first led us to showing that the Vaserstein symbol was not injective for the ring $\Gamma(S^2_{\mathbb{R}} \times S^1_{\mathbb{R}})$.

Either the Rao–van der Kallen example $\Gamma(S^3_{\mathbb{R}})$, or our example of $\Gamma(S^2_{\mathbb{R}} \times S^1_{\mathbb{R}})$, shows that the coordinate ring of the algebraic orthogonal group would also give an example of a smooth real affine threefold for which the Vaserstein symbol is not injective. R.A. Rao asked if there was an uncountable collection of principal basic open subsets of this variety which were non-isomorphic. M.V. Nori felt that this should be possible to show, and moreover one could (and should) also get such a collection of real varieties **which contain all the real points**.

This would then give *an uncountable family of smooth birational real threefolds* for which the Vaserstein symbol is not injective. Our main result Theorem 6.3 gives a countable family of such examples.